Representations of Fibonacci and Riemann's zeta function

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We define the fibonacci sequence φ_n as follows

Definition 1. *Let* $n \in \mathbb{N}$

$$\begin{aligned}\varphi_0 &= 0\\ \varphi_1 &= 1\\ \varphi_n &= \varphi_{n-1} + \varphi_{n-2}\end{aligned}$$

We have proved the following theorems

Theorem 1 (Limit A)**.** *Let* $k, m, r \in \mathbb{N}$, $m \geqslant 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} = \frac{\varphi_r x^r}{x^{2r} - \beta_r x^r + 1} \tag{1}$$

Theorem 2 (Limit B)**.** *Let* $k, m, r \in \mathbb{N}$, $m > 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\sum_{k=0}^{\infty} \varphi_{rk+\frac{r}{2}} x^{rk} = \frac{\varphi_{\frac{r}{2}}(x^r + 1)}{x^{2r} - \beta_r x^r + 1} \tag{2}$$

And ask about $\Theta(x, r)$ with $x, r \in \mathbb{R}$

$$\Theta(x, r) = \frac{x^r}{x^{2r} - \beta_r x^r + 1} \tag{3}$$

So we have built the integral of $\varphi(x, r)$ in the following way

Theorem 3. *Let* $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 \leqslant x \leqslant 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\int \varphi(x, r) x^r dr = \Upsilon(r)\, \Theta(x, r) \tag{4}$$

with the condition of $\varphi(1, r) = \varphi_r$, where $\Upsilon(r)$ is defined as follows

$$\Upsilon(r) = \frac{(2 - \beta(r))}{\sqrt{5}} \Big(\frac{\psi^r}{\log \psi} - \frac{\tau^r}{\log \tau} \Big), \tag{5}$$

let $\psi = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$, then $\beta(r)$ is defined as

$$\beta(r) = \psi^r + \tau^r \tag{6}$$

and φ_n is

$$\varphi_n = \frac{\psi^n - \tau^n}{\sqrt{5}}. \tag{7}$$

And with the idea of the following

$$\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk} = \Theta(x, r) \tag{8}$$

that it's always true for all $r = 2^m \geqslant 4$, where $m \in \mathbb{N}$, we ask for all of the instances that generate $\Theta(x, r)$.

So we ended up with this theorem

Theorem 4 (Representation A)**.** *Let* $j, k, n, s, t, u_k, v, v_j, w, z, \tau_k^j \in \mathbb{N},\ x, r \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,$ $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}, \quad t = \sum u_k, \quad s = \sum v_k, \quad z = 2s + w + u_0$

$$\varphi_r^z \left[\prod_{k=1}^{n} \left(\varphi_r^{(k)}\right)^{u_k}\right] \left(\int \varphi_r \, dr\right)^v \left[\prod_{j=0}^{n} \left(\sum_{k=0}^{j} \tau_k^j \, (\log x)^{j-k} \, \varphi(x,r)^{(k)}\right)^{v_j}\right] \left(\int \varphi(x,r) x^r dr\right)^w \left(2 - \beta(r)\right)^{2t+v} = \Upsilon(r)^{v+w} \left[\prod_{k=0}^{n} \left(\chi_k(1,r)\right)^{u_k}\right] \left[\prod_{j=0}^{n} \left(\chi_j(x,r)\right)^{v_j}\right] \left(\frac{1}{x^{rs}}\right) \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{2s+w} \tag{9}$$

where $\varphi(x,r)^{(k)}$ is the k-derivative.
Then we did the same for the following equation

$$\int \varphi_n(x,r) x^r dr = \Upsilon_n(r) \, \Theta^n(x,r) \tag{10}$$

And ended up with something similar

Theorem 5 (Representation B)**.** *Let* $j, k, n, s, t, u_k, v, v_j, w, z, \tau_k^j \in \mathbb{N},\ x, r \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,$ $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}, \quad s = \sum v_k, \quad t = \sum u_k, \quad z = (n+1)s + nw + u_0$

$$\varphi_r^z \left[\prod_{k=1}^{i} \left(\varphi_r^{(k)}\right)^{u_k}\right] \left(\int \varphi_r \, dr\right)^v \left[\prod_{j=0}^{i} \left(\sum_{k=0}^{j} \tau_k^j \, (\log x)^{j-k} \, \varphi_n(x,r)^{(k)}\right)^{v_j}\right] \left(\int \varphi_n(x,r) x^r dr\right)^w \left(2 - \beta(r)\right)^{(n+1)t+nv} = \Upsilon(r)^v \, \Upsilon_n(r)^w \left[\prod_{k=0}^{i} \left(\chi_k(1,r,n)\right)^{u_k}\right] \left[\prod_{j=0}^{i} \left(\chi_j(x,r,n)\right)^{v_j}\right] \left(\frac{1}{x^{rs}}\right) \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{(n+1)s+nw} \tag{11}$$

where $\chi(x,r,n)$ is define as follows

Definition 2.

$$\Xi(x,r,n) = \chi_0(x,r,n) = n \, \Upsilon_n(r) \Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big) + (\Upsilon_n(r))' \Big(x^r + \frac{1}{x^r} - \beta(r)\Big) \tag{12}$$

$$\chi_{j+1}(x,r,n) = (\chi_j(x,r,n))' + (n+1) \left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right) \Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big) \chi_j(x,r,n) \tag{13}$$

Next we try to develop the quotient, so we define the following

Definition 3.

$$\rho(x,r,n,s) = \left[\frac{1}{\prod_{j=1}^{s} \varphi_r^{(j)}}\right] \prod_{j=1}^{s} \sum_{k=0}^{j} \tau_k^j \, (\log x)^{j-k} \, \varphi_n(x,r)^{(k)} \left(\int \varphi_n(x,r) x^r dr\right)^{-s} \tag{14}$$

$$\tau(x,r,n,s) = \left[\frac{1}{\prod_{j=1}^{s} \chi_j(1,r,n)}\right] \frac{\prod_{j=1}^{s} \chi_j(x,r,n)}{x^{rs}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^s \tag{15}$$

And we get this

Theorem 6 (Representation C)**.** *Let* $n, s \in \mathbb{N},\ x, r \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{1}{2 - \beta(r)}\right]^s \left(\int \varphi_r \, dr\right)^s \rho(x,r,n,s) = \tau(x,r,n,s) \tag{16}$$

And with the help of some substitutions we get this one

Theorem 7 (Representation H). *Let* $s, u, v \in \mathbb{N},\ x, r \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{(2-\beta(r))^{u-1}\,(2+\beta(r))^{v+1}}{\Upsilon(r)^u\,\Gamma(r)^{v+1}} \right]^s \left(\int \varphi_r\,dr \right)^{(u+1)s} \left(\int \beta_r\,dr \right)^{(v+1)s} \rho(x, r, 1, s) = \tau(x, r, 1, s) \tag{17}$$

where $\Gamma(r)$ is define as follows

$$\Gamma(r) = (2+\beta(r)) \int \beta_r\,dr \tag{18}$$

Finally, if we separate the x part from the r part we get

Theorem 8 (Formula 1). *Let* $j, k, n, \tau_k^j \in \mathbb{N},\ x, r, z \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^z = \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^z \left[\frac{\chi_j(x, r, n)\ \left(\int \varphi_n(x, r) x^r\,dr\right)}{x^r\,\Upsilon_n(r)\ \sum_{k=0}^{j} \tau_k^j\,(\log x)^{j-k}\,\varphi_n(x, r)^{(k)}} \right]^z, \tag{19}$$

and for the r part we get

Theorem 9 (Formula 2). *Let* $u, v \in \mathbb{N},\ r, z \in \mathbb{R},\ r \geqslant 4$

$$\varphi_r^z = \Xi_(1, r)^z \left[\frac{(2-\beta(r))^{u-1}\,(2+\beta(r))^{v}}{\Upsilon(r)^{u-1}\,\Gamma(r)^{v}} \right]^z \left(\int \varphi_r\,dr \right)^{(u+1)z} \left(\int \beta_r\,dr \right)^{vz} \tag{20}$$

So if we just multiply both we get φ_r^{2z} for x, r variables or the following formula

Theorem 10 (Formula 7). *Let* $k, u, v \in \mathbb{N},\ x, r, z \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^z = \Xi_(1, r)^{\frac{z}{2}} \left[\frac{(2-\beta(r))^{u-1}\,(2+\beta(r))^{v}}{\Upsilon(r)^{u}\,\Gamma(r)^{v}} \right]^{\frac{z}{2}} \left(\int \varphi_r\,dr \right)^{(u+1)\frac{z}{2}} \left(\int \beta_r\,dr \right)^{v\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{\frac{z}{2}} \left[\frac{\Xi(x, r)\ \left(\int \varphi(x, r) x^r\,dr\right)}{x^r\,\varphi(x, r)} \right]^{\frac{z}{2}} \tag{21}$$

And if we get the inverse

Theorem 11 (Formula 8). *Let* $k, u, v \in \mathbb{N},\ x, r, z \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\frac{1}{\varphi_r^z} = \left[\frac{\Upsilon(r)^u\,\Gamma(r)^v}{(2-\beta(r))^{u-1}\,(2+\beta(r))^{v}\,\Xi_(1, r)} \right]^{\frac{z}{2}} \left(\int \varphi_r\,dr \right)^{-(u+1)\frac{z}{2}} \left(\int \beta_r\,dr \right)^{-v\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{-\frac{z}{2}} \left[\frac{x^r\,\varphi(x, r)}{\Xi(x, r)\ \left(\int \varphi(x, r) x^r\,dr\right)} \right]^{\frac{z}{2}} \tag{22}$$

We can perform the sum and we get

Theorem 12. *Let* $k, u, v \in \mathbb{N},\ x, r, z \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\zeta_r(z) = 1 + \frac{1}{2^z} + \sum_{r=4}^{\infty} \frac{1}{\varphi_r^z} = 1 + \frac{1}{2^z} + \sum_{r=4}^{\infty} \left[\frac{\Upsilon(r)^u\,\Gamma(r)^v}{(2-\beta(r))^{u-1}\,(2+\beta(r))^{v}\,\Xi_(1, r)} \right]^{\frac{z}{2}} \left(\int \varphi_r\,dr \right)^{-(u+1)\frac{z}{2}} \left(\int \beta_r\,dr \right)^{-v\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{-\frac{z}{2}} \left[\frac{x^r\,\varphi(x, r)}{\Xi(x, r)\ \left(\int \varphi(x, r) x^r\,dr\right)} \right]^{\frac{z}{2}} \tag{23}$$

this is always true for all r that are power of 2.
But we can ask for all $s \in \mathbb{R}$ that are between $2^{m-1} < s \leqslant 2^m$, where $r = 2^m$, we get this

$$\zeta_s(z) = 1 + \frac{1}{2^z} + \sum_{s=4}^{\infty} \frac{1}{\varphi_s^z} = 1 + \frac{1}{2^z} + \sum_{s=4}^{\infty} \left[\frac{\Upsilon(r)\Upsilon(s)^{u-1}\,\Gamma(s)^v}{(2-\beta(s))^{u-1}\,(2+\beta(s))^{v}\,\Xi_(1, s)} \right]^{\frac{z}{2}} \left(\int \varphi_s\,ds \right)^{-(u+1)\frac{z}{2}} \left(\int \beta_s\,ds \right)^{-v\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_s \frac{\varphi_{rk}}{\varphi_r} x^{rk} \right)^{-\frac{z}{2}} \left[\frac{x^r\,\varphi(x, r)}{\Xi(x, r)\ \left(\int \varphi(x, r) x^r\,dr\right)} \right]^{\frac{z}{2}} \tag{24}$$

and for a certain subsequence $\alpha = \{\varphi_s\}$ the following holds

$$\zeta_s^\alpha(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{25}$$

We can conclude that for all z between $0 < z < 2$

$$\zeta_s^\alpha(z) = 1 + \frac{1}{2^z}, \tag{26}$$

meaning that $\zeta_s^\alpha(z)$ generates a 0.

References

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