Representaciones de Fibonacci

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2020

A mi familia y a mis amigos.

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Introducción

Nuestra primera motivación para trabajar con la sucesión de fibonacci (φ_n), fue la de intentar calcular de una manera más eficaz la fórmula de Binet. Por lo cual nos dimos a la tarea de calcular la serie finita de los φ_{2n} de una manera constructiva de tal manera que iterando llegamos a la suma finita de los φ_{rn}, donde $r = 2^m$. Esta serie finita acumula efectivamente φ_r, teorema 37.
Además nos fijamos que sucede cuando sumamos s al índice de fibonacci, es decir φ_{n+s}, intentando de alguna manera separar esta s de forma lineal. Y en efecto, a partir de $m \geqslant 2$, sumar $s = 2^{m-1}$ al índice de fibonacci se distribuye dentro de φ_s, teorema 38. Sin embargo nuestra construcción finita únicamente nos permitía trabajar con r y s potencias de dos, por lo cual nos vimos obligados a utilizar ecuaciones diferenciales para intentar plantear el caso general utilizando nuestra evidencia para las potencias de dos.
Conocemos la maravillosa fórmula para calcular el n-ésimo número de fibonacci gracias a Binet: para todo $n \in \mathbb{N}$, con $\psi = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$, tenemos que

$$\varphi_n = \frac{\psi^n - \tau^n}{\sqrt{5}}$$

y sabemos esto gracias a la expansión en series de potencias de la función:

$$\sum_{k=0}^{\infty} \varphi_k x^k = \frac{x}{1 - x - x^2} = \frac{-x}{x^2 + x - 1}$$

¿Pero qué sucede con la suma de los números n-ésimos que aparecen en un orden par?

$$\sum_{k=0}^{\infty} \varphi_{2k} x^{2k} = \frac{x^2}{x^4 - 3x^2 + 1}$$

Y para los números n-ésimos que aparecen en un orden múltiplo de cuatro:

$$\sum_{k=0}^{\infty} \varphi_{4k} x^{4k} = \frac{3x^4}{x^8 - 7x^4 + 1}$$

Finalmente encontramos un patrón y la forma general para los múltiplos de r

$$\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} = \frac{\varphi_r x^r}{x^{2r} - \beta_r x^r + 1},$$

donde r es una potencia de 2 y β está definida de la siguiente manera:

$$\begin{aligned} \beta_1 &= 1 \\ \beta_2 &= 3 \\ \beta_{2^n} &= \beta_{2^{n-1}}^2 - 2, \end{aligned}$$

o bien,

$$\beta_r = \psi^r + \tau^r = \left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{1-\sqrt{5}}{2}\right)^r.$$

1 Sumando y alternando

1.1 Fibonacci sobre los naturales

Definición 1. *Sea $n \in \mathbb{N}$, definimos φ_n de la siguiente manera:*

$$\begin{aligned}\varphi_0 &= 0\\ \varphi_1 &= 1\\ \varphi_n &= \varphi_{n-1} + \varphi_{n-2}\end{aligned}$$

Teorema 1. *Sea $k, n \in \mathbb{N}$*

$$\sum_{k=0}^{n} \varphi_k = \varphi_{n+2} - 1 \tag{1}$$

Demostración. Sea $n \in \mathbb{N}$, tenemos que

$$\varphi_n + \varphi_{n+1} = \varphi_{n+2}$$

y podemos aplicar la suma en ambos lados de la igualdad:

$$\begin{aligned}\sum_{k=0}^{n} \varphi_k + \sum_{k=1}^{n+1} \varphi_k &= \sum_{k=2}^{n+2} \varphi_k\\ \sum_{k=0}^{n} \varphi_k + \Big(\varphi_1 + \sum_{k=2}^{n+1} \varphi_k\Big) &= \Big(\sum_{k=2}^{n+1} \varphi_k + \varphi_{n+2}\Big)\\ \sum_{k=0}^{n} \varphi_k + \varphi_1 &= \varphi_{n+2}\\ \sum_{k=0}^{n} \varphi_k + 1 &= \varphi_{n+2}\\ \sum_{k=0}^{n} \varphi_k &= \varphi_{n+2} - 1\end{aligned}$$

■

Teorema 2. *Sea $k, n \in \mathbb{N}$, $n > 1$*

$$\sum_{k=2}^{n} (-1)^{n-k} \varphi_k = \varphi_{n-1} \tag{2}$$

Demostración. Sea $n \in \mathbb{N}$, $n > 1$
Si $n = 2$, entonces

$$\sum_{k=2}^{2} (-1)^{2-k} \varphi_k = \varphi_2 = \varphi_1 = 1$$

Y si $n > 2$, entonces usando la definición recursiva de los números de Fibonacci podemos generar la siguiente enumeración:

$$\begin{aligned}\varphi_{n-2} + \varphi_{n-1} &= \varphi_n\\ -\varphi_{n-3} - \varphi_{n-2} &= -\varphi_{n-1}\\ \varphi_{n-4} + \varphi_{n-3} &= \varphi_{n-2}\\ -\varphi_{n-5} - \varphi_{n-4} &= -\varphi_{n-3}\\ &\vdots\\ (-1)^{n-1}\varphi_1 + (-1)^{n-1}\varphi_2 &= (-1)^{n-1}\varphi_3\\ (-1)^{n}\varphi_0 + (-1)^{n}\varphi_1 &= (-1)^{n}\varphi_2\end{aligned}$$

y sumando todo esto tenemos que

$$\begin{aligned}(-1)^n\varphi_0 + \varphi_{n-1} &= \sum_{k=2}^{n}(-1)^{n+2-k}\varphi_k \\ 0 + \varphi_{n-1} &= \sum_{k=2}^{n}(-1)^2(-1)^{n-k}\varphi_k \\ \varphi_{n-1} &= \sum_{k=2}^{n}(-1)^{n-k}\varphi_k\end{aligned}$$

■

Lema 1. *Sea $k, n \in \mathbb{N}$, $n > 1$*

$$\textit{Si } n \textit{ es par, por lo tanto } \sum_{k=1}^{\frac{n}{2}}\varphi_{2k} = \varphi_{n+1} - 1 \tag{3}$$

$$\textit{Si } n \textit{ es impar, por lo tanto } \sum_{k=1}^{\frac{n-1}{2}}\varphi_{2k+1} = \varphi_{n+1} - 1 \tag{4}$$

Demostración. Sea $n \in \mathbb{N}$, $n > 1$, usando (1) y (2) tenemos que

$$\begin{aligned}\sum_{k=2}^{n}\varphi_k &= \varphi_{n+2} - 2 \\ \sum_{k=2}^{n}(-1)^{n-k}\varphi_k &= \varphi_{n-1}\end{aligned}$$

y sumando tenemos que

$$\sum_{k=2}^{n}\varphi_k + \sum_{k=2}^{n}(-1)^{n-k}\varphi_k = \varphi_{n+2} + \varphi_{n-1} - 2$$

si n es par y $n > 1$ tenemos que

$$\begin{aligned}\sum_{k=2}^{n}\varphi_k + \sum_{k=2}^{n}(-1)^{n-k}\varphi_k = 2\sum_{k=1}^{\frac{n}{2}}\varphi_{2k} &= \varphi_{n+1} + \varphi_n + \varphi_{n-1} - 2 \\ 2\sum_{k=1}^{\frac{n}{2}}\varphi_{2k} &= 2\varphi_{n+1} - 2 \\ \sum_{k=1}^{\frac{n}{2}}\varphi_{2k} &= \varphi_{n+1} - 1\end{aligned}$$

y si n es impar y $n > 1$ tenemos que

$$\begin{aligned}\sum_{k=2}^{n}\varphi_k + \sum_{k=2}^{n}(-1)^{n-k}\varphi_k = 2\sum_{k=1}^{\frac{n-1}{2}}\varphi_{2k+1} &= 2\varphi_{n+1} - 2 \\ \sum_{k=1}^{\frac{n-1}{2}}\varphi_{2k+1} &= \varphi_{n+1} - 1\end{aligned}$$

■

Teorema 3. *Sea* $k, n \in \mathbb{N}$*,* $n > 1$

$$\sum_{k=1}^{n} \varphi_{2k} = \varphi_{2n+1} - 1 \tag{5}$$

Demostración. Sea $n \in \mathbb{N}$, n par, usando (3) con $m = \frac{n}{2}$ tenemos que

$$\sum_{k=1}^{m} \varphi_{2k} = \varphi_{2m+1} - 1$$

para todo $m > 1$. ■

Teorema 4. *Sea* $k, n \in \mathbb{N}$*,* $n > 1$

$$\sum_{k=1}^{n-1} \varphi_{2k+1} = \varphi_{2n} - 1 \tag{6}$$

Demostración. Sea $n \in \mathbb{N}$, n impar, usando (4) con $m - 1 = \frac{n-1}{2}$ tenemos que

$$\sum_{k=1}^{m-1} \varphi_{2k+1} = \varphi_{2m} - 1$$

para todo $m > 1$. ■

Corolario 1. *Sea* $k, n \in \mathbb{N}$*,* $n > 0$

$$\sum_{k=0}^{n-1} \varphi_{2k+1} = \varphi_{2n} \tag{7}$$

Demostración. Sea $n \in \mathbb{N}$, $n > 0$
Si $n = 1$ tenemos que

$$\sum_{k=0}^{0} \varphi_{2k+1} = \varphi_1 = \varphi_2 = 1$$

Y si $n > 1$, usando (6) tenemos que

$$\sum_{k=1}^{n-1} \varphi_{2k+1} = \varphi_{2n} - 1$$
$$\varphi_1 + \sum_{k=1}^{n-1} \varphi_{2k+1} = \varphi_{2n} - 1 + \varphi_1$$
$$\sum_{k=0}^{n-1} \varphi_{2k+1} = \varphi_{2n} - 1 + 1$$
$$\sum_{k=0}^{n-1} \varphi_{2k+1} = \varphi_{2n}$$

para todo $n > 1$. ■

1.2 Fibonacci sobre los negativos

Ahora bien usando la siguiente enumeración

$$\varphi_1 = 1, \quad \varphi_0 = 0, \quad \varphi_{-1} = 1, \quad \varphi_{-2} = -1, \quad \varphi_{-3} = 2$$

podemos calcular φ_n sobre los negativos y entonces podemos definir φ_n sobre los enteros.

Definición 2. *Sea $n \in \mathbb{Z}$,*

$$\begin{aligned}\varphi_0 &= 0,\\ \varphi_1 &= 1,\\ \varphi_n &= \varphi_{n-1} + \varphi_{n-2}\end{aligned}$$

Además podemos notar que si n es par entonces

$$\varphi_{-n} = -\varphi_n$$

y si n es impar

$$\varphi_{-n} = \varphi_n$$

Teorema 5. *Sea $k, n \in \mathbb{N}$*

$$\sum_{k=0}^{n} \varphi_{-k} = 1 + (-1)^{n-1}\varphi_{n-1} \tag{8}$$

Demostración. Sea $n \in \mathbb{N}$, entonces tenemos

$$\varphi_{-n} = (-1)^{n-1}\varphi_n$$

y si sumamos desde 1 hasta n tenemos que

$$\begin{aligned}\sum_{k=1}^{n} \varphi_{-k} &= \sum_{k=1}^{n}(-1)^{k-1}\varphi_k\\ &= (-1)\sum_{k=1}^{n}(-1)^{k}\varphi_k\\ &= (-1) * (-1)^n \sum_{k=1}^{n}(-1)^{k-n}\varphi_k\\ &= (-1)^{n-1}\sum_{k=1}^{n}(-1)^{n-k}\varphi_k\\ &= (-1)^{n-1}\Big(\sum_{k=2}^{n}(-1)^{n-k}\varphi_k + (-1)^{n-1}\varphi_1\Big)\\ &= (-1)^{n-1}\Big(\sum_{k=2}^{n}(-1)^{n-k}\varphi_k\Big) + \varphi_1\end{aligned}$$

Y usando (2) tenemos que

$$\begin{aligned}\varphi_{-0} + \sum_{k=1}^{n} \varphi_{-k} &= (-1)^{n-1}\varphi_{n-1} + \varphi_1\\ \sum_{k=0}^{n} \varphi_{-k} &= (-1)^{n-1}\varphi_{n-1} + 1\end{aligned}$$

para todo $n > 1$. ∎

Lema 2. *Sea* $n \in \mathbb{N}$, $k \in \mathbb{Z}$

$$\sum_{k=-n}^{n} \varphi_k = \varphi_{n+2} + (-1)^{n-1}\varphi_{n-1} \tag{9}$$

Demostración. Sea $n \in \mathbb{N}$
Si $n = 0$, tenemos que

$$\begin{aligned}\sum_{k=-0}^{0} \varphi_k &= \varphi_0 = \varphi_2 - \varphi_{-1} \\ &= 0 = 1 - 1\end{aligned}$$

Y si $n > 0$, tenemos que

$$\begin{aligned}\sum_{k=-n}^{n} \varphi_k &= \sum_{k=1}^{n} \varphi_{-k} + \varphi_0 + \sum_{k=1}^{n} \varphi_k \\ &= \sum_{k=1}^{n} \varphi_{-k} + 0 + \sum_{k=1}^{n} \varphi_k \\ &= \sum_{k=1}^{n} \varphi_{-k} + \sum_{k=1}^{n} \varphi_k\end{aligned}$$

Usando (1) y (8) resulta que

$$\begin{aligned}\sum_{k=-n}^{n} \varphi_k &= (1 + (-1)^{n-1}\varphi_{n-1}) + (\varphi_{n+2} - 1) \\ &= (-1)^{n-1}\varphi_{n-1} + \varphi_{n+2}\end{aligned}$$

para todo $n > 0$. ∎

Corolario 2. *Sea* $n \in \mathbb{N}$, $k \in \mathbb{Z}$

$$\textit{Si } n \textit{ es par, por lo tanto } \sum_{k=-n}^{n} \varphi_k = 2\varphi_n \tag{10}$$

$$\textit{Si } n \textit{ es impar, por lo tanto } \sum_{k=-n}^{n} \varphi_k = 2\varphi_{n+1} \tag{11}$$

Demostración. Sea $n \in \mathbb{N}$
Si n es par y usando (9) tenemos que

$$\begin{aligned}\sum_{k=-n}^{n} \varphi_k &= \varphi_{n+2} + (-1)^{n-1}\varphi_{n-1} \\ &= \varphi_{n+2} - \varphi_{n-1} = \varphi_{n+1} + \varphi_n - \varphi_{n-1} \\ &= \varphi_{n+1} + \varphi_{n-1} + \varphi_{n-2} - \varphi_{n-1} = \varphi_{n+1} + \varphi_{n-2} \\ &= \varphi_n + \varphi_{n-1} + \varphi_{n-2} = \varphi_n + \varphi_n \\ &= 2\varphi_n\end{aligned}$$

Y si n es impar y usando (9) tenemos que

$$\begin{aligned}\sum_{k=-n}^{n} \varphi_k &= \varphi_{n+2} + (-1)^{n-1}\varphi_{n-1} \\ &= \varphi_{n+2} + \varphi_{n-1} = \varphi_{n+1} + \varphi_n + \varphi_{n-1} \\ &= \varphi_{n+1} + \varphi_{n+1} \\ &= 2\varphi_{n+1}\end{aligned}$$

∎

Corolario 3. *Sea* $n \in \mathbb{N}$

$$\varphi_n = \frac{\varphi_{n+2} - \varphi_{n-1}}{2} \tag{12}$$

$$\varphi_n = \frac{\varphi_{n+1} + \varphi_{n-2}}{2} \tag{13}$$

Demostración. Sea $n \in \mathbb{N}$
Si $n = 0$ tenemos que

$$\varphi_0 = \frac{\varphi_2 - \varphi_{-1}}{2} = \frac{\varphi_1 + \varphi_{-2}}{2}$$
$$0 = \frac{1-1}{2} = \frac{1+(-1)}{2}$$

Si n es par y usando (9) y (10) tenemos que

$$\varphi_{n+2} - \varphi_{n-1} = 2\varphi_n$$
$$\frac{\varphi_{n+2} - \varphi_{n-1}}{2} = \varphi_n$$

para todo $n \in \mathbb{N}$
Entonces $n-1$ es impar y usando (9) y (10) tenemos que

$$\varphi_{n+1} + \varphi_{n-2} = 2\varphi_n$$
$$\frac{\varphi_{n+1} + \varphi_{n-2}}{2} = \varphi_n$$

para todo $n \in \mathbb{N}$
Y si n es impar y usando (9) y (10) tenemos que

$$\varphi_{n+2} + \varphi_{n-1} = 2\varphi_{n+1}$$
$$\frac{\varphi_{n+2} + \varphi_{n-1}}{2} = \varphi_{n+1}$$

para todo $n \in \mathbb{N}$, si hacemos n'=n+1, resulta que

$$\frac{\varphi_{n'+1} + \varphi_{n'-2}}{2} = \varphi_{n'}$$

para todo $n \in \mathbb{N}$.
Entonces $n-1$ es par y usando (9) y (10) tenemos que

$$\varphi_{n+1} - \varphi_{n-2} = 2\varphi_{n-1}$$
$$\frac{\varphi_{n+1} - \varphi_{n-2}}{2} = 2\varphi_{n-1}$$

para todo $n \in \mathbb{N}$, si hacemos n'=n-1, resulta que

$$\frac{\varphi_{n'+2} - \varphi_{n'-1}}{2} = 2\varphi_{n'}$$

para todo $n > 0$. ∎

2 Series de potencias

2.1 Funciones generadoras

Empezaremos primeramente por construir la suma finita de los φ_k alrededor de su serie de potencias, usaremos el rango de x entre 0 y 1, y pediremos únicamente que $x \neq \frac{-1+\sqrt{5}}{2}$ para que el denominador sea diferente de 0.

Teorema 6. *Sea $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$*

$$\sum_{k=0}^{n} \varphi_k x^k = \frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2 + x - 1} \tag{14}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$
Si $n = 0$ entonces

$$\sum_{k=0}^{0} \varphi_k x^k = \frac{\varphi_0 x^2 + \varphi_1 x - x}{x^2 + x - 1}$$
$$0 = \varphi_0 x^0 = \frac{x - x}{x^2 + x - 1} = 0$$

Si $n = 1$ entonces

$$\sum_{k=0}^{1} \varphi_k x^k = \frac{\varphi_1 x^3 + \varphi_2 x^2 - x}{x^2 + x - 1}$$
$$x = \varphi_0 x^0 + \varphi_1 x = \frac{x^3 + x^2 - x}{x^2 + x - 1} = x\Big(\frac{x^2 + x - 1}{x^2 + x - 1}\Big) = x$$

Y si $n > 1$, sea

$$F_n = \sum_{k=0}^{n} \varphi_k x^k$$

entonces tenemos que

$$\sum_{k=0}^{n} \varphi_k x^{k+r} = x^r * F_n(x) \tag{15}$$

$$\sum_{k=0}^{n-2} \varphi_{k+2} x^k = \frac{\frac{F_n(x) - \varphi_0}{x} - \varphi_1}{x} = \frac{\frac{F_n(x)}{x} - 1}{x} = \frac{F_n(x) - x}{x^2} \tag{16}$$

Y sabemos que

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \tag{17}$$

con esto podemos fijarnos en la siguiente suma

$$S_n(x) = F_n(x) + xF_n(x) + x^2 F_n(x) + \cdots + x^n F_n(x) \tag{18}$$

$$S_n(x) = (1 + x + x^2 + \cdots + x^n) * F_n(x)$$
$$S_n(x) = \Big(\sum_{k=0}^{n} x^k\Big) F_n(x)$$

Usando (17) tenemos

$$S_n(x) = \Big(\frac{1 - x^{n+1}}{1 - x}\Big) * F_n(x) \tag{19}$$

Además (18) se puede expresar de la siguiente forma con ayuda de (15)

$$\begin{aligned}S_n(x) =&\varphi_0 + \varphi_1 x + \varphi_2 x^2 + \cdots + \varphi_n x^n +\\ &\varphi_0 x + \varphi_1 x^2 + \varphi_2 x^3 + \cdots + \varphi_n x^{n+1} +\\ &\vdots\\ &\varphi_0 x^r + \varphi_1 x^{r+1} + \varphi_2 x^{r+2} + \cdots + \varphi_n x^{n+r} +\\ &\vdots\\ &\varphi_0 x^n + \varphi_1 x^{n+1} + \varphi_2 x^{n+2} + \cdots + \varphi_n x^{2n}\\ S_n(x) =& \sum_{r=0}^{n}\sum_{k=0}^{n} \varphi_k x^{k+r}\end{aligned}$$

Esta fórmula también se puede escribir como

$$\begin{aligned}S_n(x) =&\sum_{r=0}^{n}\Big(\sum_{i=0}^{n-r}\varphi_i\Big)x^{n-r} + \sum_{r=1}^{n}\Big(\sum_{i=r}^{n}\varphi_i\Big)x^{n+r}\\ =&\sum_{r=0}^{n}\Big(\sum_{i=0}^{n-r}\varphi_i\Big)x^{n-r} + \sum_{r=1}^{n}\Big(\sum_{i=0}^{n}\varphi_i - \sum_{i=0}^{r-1}\varphi_i\Big)x^{n+r}\end{aligned}$$

Usando (1) tenemos que

$$\begin{aligned}S_n(x) =&\sum_{r=0}^{n}(\varphi_{n-r+2} - 1)x^{n-r} + \sum_{r=1}^{n}\big((\varphi_{n+2} - 1) - (\varphi_{r+1} - 1)\big)x^{n+r}\\ =&\sum_{r=0}^{n}(\varphi_{r+2} - 1)x^{r} + x^n\sum_{r=1}^{n}\big(\varphi_{n+2} - \varphi_{r+1}\big)x^{r}\\ =&\sum_{r=0}^{n}(\varphi_{r+2} - 1)x^{r} + x^n\sum_{r=0}^{n-1}\big(\varphi_{n+2} - \varphi_{r+2}\big)x^{r+1}\\ =&\sum_{r=0}^{n}(\varphi_{r+2} - 1)x^{r} + x^{n+1}\sum_{r=0}^{n-1}\big(\varphi_{n+2} - \varphi_{r+2}\big)x^{r}\\ =&\sum_{r=0}^{n}\varphi_{r+2}x^{r} - \sum_{r=0}^{n}x^r + x^{n+1}\Big(\varphi_{n+2}\sum_{r=0}^{n-1}x^r - \sum_{r=0}^{n-1}\varphi_{r+2}x^r\Big)\end{aligned}$$

y sustituyendo con (16) y (17) resulta que

$$\begin{aligned}S_n(x) =&\Big(\frac{F_n(x) - x}{x^2} + \varphi_{n+1}x^{n-1} + \varphi_{n+2}x^n\Big) - \frac{1 - x^{n+1}}{1 - x} +\\ &x^{n+1}\Big(\varphi_{n+2}\Big(\frac{1 - x^n}{1 - x}\Big) - \Big(\frac{F_n(x) - x}{x^2} + \varphi_{n+1}x^{n-1}\Big)\Big)\end{aligned}$$

igualando con (19) resulta lo siguiente

$$\begin{aligned}\Big(\frac{1 - x^{n+1}}{1 - x}\Big)F_n(x) =&\frac{F_n(x) - x}{x^2} + \varphi_{n+1}x^{n-1} + \varphi_{n+2}x^n - \frac{1 - x^{n+1}}{1 - x} +\\ &x^{n+1}\Big(\varphi_{n+2}\Big(\frac{1 - x^n}{1 - x}\Big) - \Big(\frac{F_n(x) - x}{x^2} + \varphi_{n+1}x^{n-1}\Big)\Big)\end{aligned}$$

si hacemos $a = \varphi_{n+1}$, $b = \varphi_{n+2}$, $c = \varphi_n$ y $y = F_n(x)$ entonces tenemos que

$$\begin{aligned}\Big(\frac{1 - x^{n+1}}{1 - x} - \frac{1}{x^2} + x^{n-1}\Big)y = bx^n + ax^{n-1} - \frac{1 - x^{n+1}}{1 - x} - \frac{1}{x} +\\ x^{n+1}\Big(\frac{b(1 - x^n)}{1 - x} + \frac{1}{x} - ax^{n-1}\Big)\end{aligned}$$

$$\frac{1}{x^2}\Big(\frac{x^2-x^{n+3}}{1-x}-1+x^{n+1}\Big)y = bx^n+ax^{n-1}+\frac{-x+x^{n+2}-1+x}{x(1-x)}+$$
$$x^{n+1}\Big(\frac{x(b-bx^n)+1-x}{x(1-x)}-ax^{n-1}\Big)$$
$$\frac{1}{x^2}\Big(\frac{x^2-x^{n+3}-1+x+x^{n+1}-x^{n+2}}{1-x}\Big)y = \frac{cx^{n+1}+ax^n+(1-b)x^{n+2}-1}{x(1-x)}+$$
$$x^{n+1}\Big(\frac{bx-bx^{n+1}+1-x-ax^n+ax^{n+1}}{x(1-x)}\Big)$$
$$\frac{1}{x^2}\Big(\frac{x^{n+1}(x^2+x-1)-(x^2+x-1)}{x-1}\Big)y = \frac{(b-1)x^{n+2}-cx^{n+1}-ax^n+1}{x(x-1)}+$$
$$x^{n+1}\Big(\frac{cx^{n+1}+ax^n+(1-b)x-1}{x(x-1)}\Big)$$
$$\frac{1}{x^2}\Big(\frac{(x^{n+1}-1)(x^2+x-1)}{x-1}\Big)y = \frac{(x^{n+1}-1)(cx^{n+1}+ax^n+(1-b)x-1)}{x(x-1)}$$
$$+\frac{(b-1)x^{n+2}+(1-b)x}{x(x-1)}$$
$$\frac{1}{x^2}\Big(\frac{(x^{n+1}-1)(x^2+x-1)}{x-1}\Big)y = \frac{(x^{n+1}-1)(cx^{n+1}+ax^n+(1-b)x-1)}{x(x-1)}$$
$$-\frac{x(1-b)(x^{n+1}-1)}{x(x-1)}$$
$$\frac{1}{x^2}\Big(\frac{(x^{n+1}-1)(x^2+x-1)}{x-1}\Big)y = \frac{(x^{n+1}-1)(cx^{n+1}+ax^n-1)}{x(x-1)}$$

Ya que $0 < x < 1$ y $x \neq \frac{-1+\sqrt{5}}{2}$ podemos simplificar y resulta

$$y\Big(\frac{x^2+x-1}{x}\Big) = cx^{n+1}+ax^n-1$$

Restaurando las variables y,a,c tenemos que

$$F_n(x)\Big(\frac{x^2+x-1}{x}\Big) = \varphi_n x^{n+1}+\varphi_{n+1}x^n-1$$
$$F_n(x) = \Big(\varphi_n x^{n+1}+\varphi_{n+1}x^n-1\Big)\frac{x}{x^2+x-1}$$
$$F_n(x) = \Big(\varphi_n x^{n+2}+\varphi_{n+1}x^{n+1}-x\Big)\frac{1}{x^2+x-1}$$

Y finalmente podemos escribir

$$F_n(x) = \sum_{k=0}^{n}\varphi_k x^k = \frac{\varphi_n x^{n+2}+\varphi_{n+1}x^{n+1}-x}{x^2+x-1}$$

para todo $n > 1$ ■

Corolario 4. *Sea* $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$

$$\sum_{k=0}^{n} \varphi_{k+1} x^k = \frac{\varphi_{n+1} x^{n+2} + \varphi_{n+2} x^{n+1} - 1}{x^2 + x - 1} \tag{20}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$, usando (14) para $n+1$ tenemos

$$\begin{aligned}
\sum_{k=0}^{n+1} \varphi_k x^k &= \frac{\varphi_{n+1} x^{n+3} + \varphi_{n+2} x^{n+2} - x}{x^2 + x - 1} \\
\Big(\sum_{k=1}^{n+1} \varphi_k x^k\Big) \frac{1}{x} &= \frac{\varphi_{n+1} x^{n+2} + \varphi_{n+2} x^{n+1} - 1}{x^2 + x - 1} \\
\sum_{k=1}^{n+1} \varphi_k x^{k-1} &= \frac{\varphi_{n+1} x^{n+2} + \varphi_{n+2} x^{n+1} - 1}{x^2 + x - 1} \\
\sum_{k=0}^{n} \varphi_{k+1} x^k &= \frac{\varphi_{n+1} x^{n+2} + \varphi_{n+2} x^{n+1} - 1}{x^2 + x - 1}
\end{aligned}$$

■

Queremos preguntarnos acerca de la suma infinita por lo cual revisaremos el siguiente límite (21) y llegaremos a (22).

Teorema 7. *Sea* $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$

$$\lim_{n \to \infty} \Big(\varphi_n x^{n+1} + \varphi_{n+1} x^n - 1\Big) = -1 \tag{21}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$

$$\big|(\varphi_n x^{n+1} + \varphi_{n+1} x^n - 1) - (-1)\big| = \big|\varphi_n x^{n+1} + \varphi_{n+1} x^n\big|$$

Ya que $0 < x < 1$, podemos escribir $x = \frac{1}{y}$ para alguna $y \neq 0$

$$\big|\varphi_n x^{n+1} + \varphi_{n+1} x^n\big| = \left|\varphi_n (\frac{1}{y^{n+1}}) + \varphi_{n+1} (\frac{1}{y^n})\right|$$

Sea $\epsilon \in \mathbb{R}$, $\epsilon > 0$, $a = \varphi_{n+1}, b = y$ entonces queremos encontrar a partir de cual δ se cumple lo siguiente

$$\frac{a}{b^\delta} < \frac{\epsilon}{2}$$

Despejando δ tenemos que

$$\begin{aligned}
2a * \epsilon^{-1} &< b^\delta \\
\log(2a * \epsilon^{-1}) &< \log(b^\delta) \\
\log(2a) - \log(\epsilon) &< (\delta) \log(b) \\
\frac{\log(2a) - \log(\epsilon)}{\log(b)} &< \delta
\end{aligned}$$

con esta δ y sabiendo que $y > 0$ tenemos que

$$\frac{\varphi_n}{y^{\delta+1}} \leqslant \frac{\varphi_{n+1}}{y^{\delta+1}} < \frac{\varphi_{n+1}}{y^\delta} < \frac{\epsilon}{2}$$

con lo cual resulta que

$$\frac{\varphi_{n+1}}{y^{\delta}} < \frac{\epsilon}{2}$$
$$\frac{\varphi_{n}}{y^{\delta+1}} < \frac{\epsilon}{2}$$

y sumando obtenemos

$$\begin{aligned} 0 < \varphi_n x^{n+1} + \varphi_{n+1} x^n &< \epsilon \\ -\epsilon < \varphi_n x^{n+1} + \varphi_{n+1} x^n &< \epsilon \\ -\epsilon < \varphi_n x^{n+1} + \varphi_{n+1} x^n - 1 + 1 &< \epsilon \\ \left|(\varphi_n x^{n+1} + \varphi_{n+1} x^n - 1) - (-1)\right| &< \epsilon \end{aligned}$$

para todo $n > \delta$ ■

Teorema 8. *Sea* $k \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$

$$\sum_{k=0}^{\infty} \varphi_k x^k = \frac{x}{1-x-x^2} = \frac{-x}{x^2+x-1} \tag{22}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$, podemos usar (14)

$$\sum_{k=0}^{n} \varphi_k x^k = \frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2+x-1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned} \sum_{k=0}^{\infty} \varphi_k x^k &= \lim_{n\to\infty} \left(\frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2+x-1} \right) \\ &= \lim_{n\to\infty} (\varphi_n x^{n+1} + \varphi_{n+1} x^n - 1) \left(\frac{x}{x^2+x-1} \right) \end{aligned}$$

y aplicando (21)

$$\begin{aligned} \sum_{k=0}^{\infty} \varphi_k x^k &= \frac{-x}{x^2+x-1} \\ &= \frac{x}{1-x-x^2} \end{aligned}$$

■

Nos preguntamos acerca del siguiente límite (23) ya que éste y el Teorema (24) nos permiten calcular cada φ_n en la Fórmula de Binet (28).

Teorema 9. *Sea* $k \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$

$$\sum_{k=0}^{\infty} \varphi_{k+1} x^k = \frac{1}{1-x-x^2} \tag{23}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$, podemos usar (20)

$$\sum_{k=0}^{n} \varphi_{k+1} x^k = \frac{\varphi_{n+1} x^{n+2} + \varphi_{n+2} x^{n+1} - 1}{x^2+x-1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned}\sum_{k=0}^{\infty} \varphi_{k+1} x^k &= \lim_{n\to\infty}\Big(\frac{\varphi_{n+1}x^{n+2}+\varphi_{n+2}x^{n+1}-1}{x^2+x-1}\Big)\\ &= \lim_{n\to\infty}(\varphi_{n+1}x^{n+2}+\varphi_{n+2}x^{n+1}-1)\Big(\frac{1}{x^2+x-1}\Big)\end{aligned}$$

y aplicando (21) para $n+1$

$$\begin{aligned}\sum_{k=0}^{\infty} \varphi_{k+1} x^k &= \frac{-1}{x^2+x-1}\\ &= \frac{1}{1-x-x^2}\end{aligned}$$

■

Teorema 10. *Sea $k \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$, $\psi = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$*

$$\sum_{k=0}^{\infty} \varphi_{k+1} x^k = \frac{1}{\psi-\tau}\Big(\frac{\psi}{1-\psi x}-\frac{\tau}{1-\tau x}\Big) \tag{24}$$

Demostración. Sea $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$, $\psi = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$
Sabemos que

$$\psi * \tau = -1$$

por tanto

$$\psi = -\frac{1}{\tau} \tag{25}$$

$$\tau = -\frac{1}{\psi} \tag{26}$$

y además

$$(x+\psi)(x+\tau) = x^2+x-1 \tag{27}$$

$$\begin{aligned}\frac{1}{\psi-\tau}\Big(\frac{\psi}{1-\psi x}-\frac{\tau}{1-\tau x}\Big) &= \frac{1}{\psi-\tau}\Big(\frac{\big(\frac{-1}{\psi}\big)}{\big(\frac{-1}{\psi}\big)}\frac{\psi}{1-\psi x}-\frac{\big(\frac{-1}{\tau}\big)}{\big(\frac{-1}{\tau}\big)}\frac{\tau}{1-\tau x}\Big)\\ &= \frac{1}{\psi-\tau}\Big(\frac{-1}{x-\frac{1}{\psi}}-\frac{-1}{x-\frac{1}{\tau}}\Big)\end{aligned}$$

usando (25) y (26)

$$\begin{aligned}&= \frac{1}{\psi-\tau}\Big(\frac{-1}{x+\tau}-\frac{-1}{x+\psi}\Big)\\ &= \frac{1}{\psi-\tau}\Big(\frac{1}{x+\psi}-\frac{1}{x+\tau}\Big)\\ &= \frac{1}{\psi-\tau}\Big(\frac{(x+\tau)-(x+\psi)}{(x+\psi)(x+\tau)}\Big)\end{aligned}$$

usando (27)

$$\begin{aligned}&= \frac{1}{\psi-\tau}\Big(\frac{\tau-\psi}{x^2+x-1}\Big)\\ &= \frac{\tau-\psi}{\psi-\tau}\Big(\frac{1}{x^2+x-1}\Big)\\ &= \frac{-1}{x^2+x-1}\\ &= \frac{1}{1-x-x^2}\end{aligned}$$

■

Teorema 11 (Fórmula de Binet)**.** *Sea* $n \in \mathbb{N}$, $\psi = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$

$$\varphi_n = \frac{\psi^n - \tau^n}{\sqrt{5}} \tag{28}$$

Demostración. Sea $n \in \mathbb{N}$
Si $n = 0$

$$0 = \varphi_0 = \frac{\psi^0 - \tau^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$

Si $n > 0$, sea $x \in \mathbb{R}$, $0 < x < 1$, $x \neq \frac{-1+\sqrt{5}}{2}$, sabemos que

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{29}$$

Usando (29) con $x = \psi x$, tenemos que

$$\sum_{k=0}^{\infty} \psi^k x^k = \frac{1}{1-\psi x}$$

y si multiplicamos por ψ tenemos

$$\sum_{k=0}^{\infty} \psi^{k+1} x^k = \frac{\psi}{1-\psi x}$$

hacemos lo mismo para $x = \tau x$ y tenemos

$$\sum_{k=0}^{\infty} \tau^{k+1} x^k = \frac{\tau}{1-\tau x}$$

hacemos la resta

$$\sum_{k=0}^{\infty} (\psi^{k+1} - \tau^{k+1}) x^k = \frac{\psi}{1-\psi x} - \frac{\tau}{1-\tau x}$$

y dividimos entre $\psi - \tau$

$$\sum_{k=0}^{\infty} \frac{(\psi^{k+1} - \tau^{k+1})}{\psi - \tau} x^k = \frac{1}{\psi - \tau}\Big(\frac{\psi}{1-\psi x} - \frac{\tau}{1-\tau x}\Big)$$

Usando (24) tenemos que

$$\sum_{k=0}^{\infty} \frac{(\psi^{k+1} - \tau^{k+1})}{\psi - \tau} x^k = \sum_{k=0}^{\infty} \varphi_{k+1} x^k$$

por tanto si hacemos $k + 1 = k'$, obtenemos

$$\varphi_{k'} = \frac{(\psi^{k'} - \tau^{k'})}{\psi - \tau} = \frac{(\psi^{k'} - \tau^{k'})}{\sqrt{5}}$$

para todo $k' > 0$. ∎

Ahora construiremos la serie alternante (30) que nos servirá para generar más series (46), (47), (49) y (50). Aquí el rango de x va de -1 a 0 ya que estamos utilizando los teoremas anteriores y estamos realizando la siguiente sustitución $x = -x$, además pedimos que $x \neq \frac{1-\sqrt{5}}{2}$.

Teorema 12. *Sea* $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$

$$\sum_{k=0}^{n} (-1)^k \varphi_k x^k = \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1} \tag{30}$$

Demostración. Suponemos (14) para $x = -x$
Sea $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$

$$\begin{aligned}\sum_{k=0}^{n} (-1)^k \varphi_k x^k &= \frac{(-1)^{n+2} \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1} \\ &= \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1}\end{aligned}$$

para todo $n \in \mathbb{N}$ ∎

Teorema 13. *Sea* $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$

$$\lim_{n \to \infty} \Big((-1)^n \varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n + 1 \Big) = 1 \tag{31}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$

$$\left| ((-1)^n \varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n + 1) - (1) \right| = \left| (-1)^n \varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n \right|$$

Ya que $0 < x < 1$, podemos escribir $x = \frac{1}{y}$ para alguna $y \neq 0$

$$\left| (-1)^n \varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n \right| = \left| (-1)^n \varphi_n \frac{1}{y^{n+1}} + (-1)^{n+1} \varphi_{n+1} \frac{1}{y^n} \right|$$

Sea $\epsilon \in \mathbb{R}$, $\epsilon > 0$, $a = \varphi_{n+1}, b = y$ entonces queremos encontrar a partir de cual δ se cumple lo siguiente

$$\frac{a}{b^\delta} < \frac{\epsilon}{2}$$

ya que multiplicando por (-1) resulta

$$-\frac{\epsilon}{2} < -\frac{a}{b^\delta}$$

y uniendo estas dos desigualdades tenemos que

$$-\frac{\epsilon}{2} < (-1)^n \frac{a}{b^\delta} < \frac{\epsilon}{2}$$

Entonces despejando δ tenemos que

$$\begin{aligned} 2a * \epsilon^{-1} &< b^\delta \\ \log(2a * \epsilon^{-1}) &< \log(b^\delta) \\ \log(2a) - \log(\epsilon) &< (\delta) \log(b) \\ \frac{\log(2a) - \log(\epsilon)}{\log(b)} &< \delta \end{aligned}$$

con esta δ y sabiendo que $y > 0$ tenemos que

$$-\frac{\epsilon}{2} < (-1)^{n+1} \frac{\varphi_n}{y^{\delta+1}} \leqslant (-1)^{n+1} \frac{\varphi_{n+1}}{y^{\delta+1}} < (-1)^{n+1} \frac{\varphi_{n+1}}{y^\delta} < \frac{\epsilon}{2}$$

y además

$$-\frac{\epsilon}{2} < (-1)^n \frac{\varphi_n}{y^{\delta+1}} \leqslant (-1)^n \frac{\varphi_{n+1}}{y^{\delta+1}} < (-1)^n \frac{\varphi_{n+1}}{y^{\delta}} < \frac{\epsilon}{2}$$

con lo cual resulta que

$$\begin{aligned} -\frac{\epsilon}{2} < (-1)^{n+1} \frac{\varphi_{n+1}}{y^{\delta}} < \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} < (-1)^n \frac{\varphi_n}{y^{\delta+1}} < \frac{\epsilon}{2} \end{aligned}$$

y sumando obtenemos

$$\begin{aligned} -\epsilon < (-1)^n \varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n < \epsilon \\ -\epsilon < (-1)^n \varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n + 1 - 1 < \epsilon \\ \left| ((-1)^n \varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n + 1) - (1) \right| < \epsilon \end{aligned}$$

para todo $n > \delta$ ■

Teorema 14. *Sea $k \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$*

$$\sum_{k=0}^{\infty} (-1)^k \varphi_k x^k = \frac{x}{x^2 - x - 1} \tag{32}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$, podemos usar (30)

$$\sum_{k=0}^{n} (-1)^k \varphi_k x^k = \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k \varphi_k x^k &= \lim_{n \to \infty} \left(\frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1} \right) \\ &= \lim_{n \to \infty} \Big((-1)^n \varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n + 1 \Big) \frac{x}{x^2 - x - 1} \end{aligned}$$

y aplicando (31)

$$= \frac{x}{x^2 - x - 1}$$

■

Ya habíamos visto en (8) que la suma alternada generaba la suma de los negativos, aquí tenemos su versión en serie de potencias (33), (34).

Teorema 15. *Sea $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$*

$$\sum_{k=0}^{n} \varphi_{-k} x^k = \frac{(-1)^{n+1} \varphi_n x^{n+2} + (-1)^n \varphi_{n+1} x^{n+1} - x}{x^2 - x - 1} \tag{33}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$, podemos usar (30)

$$\sum_{k=0}^{n} (-1)^k \varphi_k x^k = \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1}$$

y multiplicando por (-1) obtenemos

$$\sum_{k=0}^{n} \varphi_{-k} x^k = \sum_{k=0}^{n} (-1)^{k+1} \varphi_k x^k = \frac{(-1)^{n+1}\varphi_n x^{n+2} + (-1)^n \varphi_{n+1} x^{n+1} - x}{x^2 - x - 1}$$

■

Teorema 16. *Sea* $k \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$

$$\sum_{k=0}^{\infty} \varphi_{-k} x^k = \frac{x}{1 + x - x^2} \tag{34}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$, podemos usar (33)

$$\sum_{k=0}^{n} \varphi_{-k} x^k = \frac{(-1)^{n+1}\varphi_n x^{n+2} + (-1)^n \varphi_{n+1} x^{n+1} - x}{x^2 - x - 1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned}
\sum_{k=0}^{\infty} \varphi_{-k} x^k &= \lim_{n\to\infty} \left(\frac{(-1)^{n+1}\varphi_n x^{n+2} + (-1)^n \varphi_{n+1} x^{n+1} - x}{x^2 - x - 1} \right) \\
&= \lim_{n\to\infty} \Big((-1)^{n+1}\varphi_n x^{n+1} + (-1)^n \varphi_{n+1} x^n - 1 \Big) \frac{x}{x^2 - x - 1} \\
&= \lim_{n\to\infty} \Big((-1)^{n}\varphi_n x^{n+1} + (-1)^{n+1} \varphi_{n+1} x^n + 1 \Big) \frac{-x}{x^2 - x - 1}
\end{aligned}$$

y aplicando (31)

$$\begin{aligned}
&= \frac{-x}{x^2 - x - 1} \\
&= \frac{x}{1 + x - x^2}
\end{aligned}$$

■

Teorema 17. *Sea* $k \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$

$$\sum_{k=0}^{\infty} \varphi_{-(k+1)} x^k = \frac{1}{1 + x - x^2} \tag{35}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$, podemos usar (34)

$$\sum_{k=0}^{\infty} \varphi_{-k} x^k = \frac{x}{1 + x - x^2}$$

y podemos dividir entre x

$$\begin{aligned}
\sum_{k=0}^{\infty} \varphi_{-k} x^{k-1} &= \frac{1}{1 + x - x^2} \\
\sum_{k=-1}^{\infty} \varphi_{-(k+1)} x^k &= \frac{1}{1 + x - x^2} \\
\varphi_{-0} x^{-1} + \sum_{k=0}^{\infty} \varphi_{-(k+1)} x^k &= \frac{1}{1 + x - x^2} \\
0 + \sum_{k=0}^{\infty} \varphi_{-(k+1)} x^k &= \frac{1}{1 + x - x^2} \\
\sum_{k=0}^{\infty} \varphi_{-(k+1)} x^k &= \frac{1}{1 + x - x^2}
\end{aligned}$$

■

Teorema 18. *Sea* $k \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$, $\sigma = \frac{-1+\sqrt{5}}{2}$, $\upsilon = \frac{-1-\sqrt{5}}{2}$

$$\sum_{k=0}^{\infty} \varphi_{-(k+1)} x^k = \frac{1}{\sigma - \upsilon}\Big(\frac{\sigma}{1-\sigma x} - \frac{\upsilon}{1-\upsilon x}\Big) \tag{36}$$

Demostración. Sea $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$, $\sigma = \frac{-1+\sqrt{5}}{2}$, $\upsilon = \frac{-1-\sqrt{5}}{2}$
Sabemos que

$$\sigma * \upsilon = -1$$

por tanto

$$\sigma = -\frac{1}{\upsilon} \tag{37}$$

$$\upsilon = -\frac{1}{\sigma} \tag{38}$$

y además

$$(x+\sigma)(x+\upsilon) = x^2 - x - 1 \tag{39}$$

$$\begin{aligned}
\frac{1}{\sigma-\upsilon}\Big(\frac{\sigma}{1-\sigma x} - \frac{\upsilon}{1-\upsilon x}\Big) &= \frac{1}{\sigma-\upsilon}\Big(\frac{\left(\frac{-1}{\sigma}\right)}{\left(\frac{-1}{\sigma}\right)}\frac{\sigma}{1-\sigma x} - \frac{\left(\frac{-1}{\upsilon}\right)}{\left(\frac{-1}{\upsilon}\right)}\frac{\upsilon}{1-\upsilon x}\Big)\\
&= \frac{1}{\sigma-\upsilon}\Big(\frac{-1}{x-\frac{1}{\sigma}} - \frac{-1}{x-\frac{1}{\upsilon}}\Big)\\
&\text{usando (37) y (38)}\\
&= \frac{1}{\sigma-\upsilon}\Big(\frac{-1}{x+\upsilon} - \frac{-1}{x-\sigma}\Big)\\
&= \frac{1}{\sigma-\upsilon}\Big(\frac{1}{x+\sigma} - \frac{1}{x+\upsilon}\Big)\\
&= \frac{1}{\sigma-\upsilon}\Big(\frac{(x+\upsilon)-(x+\sigma)}{(x+\sigma)(x+\upsilon)}\Big)\\
&\text{usando (39)}\\
&= \frac{1}{\sigma-\upsilon}\Big(\frac{\upsilon-\sigma}{x^2-x-1}\Big)\\
&= \frac{\upsilon-\sigma}{\sigma-\upsilon}\Big(\frac{1}{x^2-x-1}\Big)\\
&= \frac{-1}{x^2-x-1} = \frac{1}{1+x-x^2}\\
&\text{y usando (35)}\\
&= \sum_{k=0}^{\infty} \varphi_{-(k+1)} x^k
\end{aligned}$$

■

Aquí tenemos la fórmula para los φ_{-n}.

Teorema 19. *Sea* $n \in \mathbb{N}$, $\sigma = \frac{-1+\sqrt{5}}{2}$, $\upsilon = \frac{-1-\sqrt{5}}{2}$

$$\varphi_{-n} = \frac{\sigma^n - \upsilon^n}{\sqrt{5}} \tag{40}$$

Demostración. Sea $n \in \mathbb{N}$
Si $n = 0$

$$0 = \varphi_{-0} = \frac{\sigma^0 - \upsilon^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$

Si $n > 0$, sea $x \in \mathbb{R}$, $-1 < x < 0$, $x \neq \frac{1-\sqrt{5}}{2}$, sabemos que

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{41}$$

Usando (41) con $x = \sigma x$, tenemos que

$$\sum_{k=0}^{\infty} \sigma^k x^k = \frac{1}{1-\sigma x}$$

y si multiplicamos por σ tenemos

$$\sum_{k=0}^{\infty} \sigma^{k+1} x^k = \frac{\sigma}{1-\sigma x}$$

hacemos lo mismo para $x = \upsilon x$ y tenemos

$$\sum_{k=0}^{\infty} \upsilon^{k+1} x^k = \frac{\upsilon}{1-\upsilon x}$$

hacemos la resta

$$\sum_{k=0}^{\infty} (\sigma^{k+1} - \upsilon^{k+1}) x^k = \frac{\sigma}{1-\sigma x} - \frac{\upsilon}{1-\upsilon x}$$

y dividimos entre $\sigma - \upsilon$

$$\sum_{k=0}^{\infty} \frac{(\sigma^{k+1} - \upsilon^{k+1})}{\sigma - \upsilon} x^k = \frac{1}{\sigma - \upsilon}\Big(\frac{\sigma}{1-\sigma x} - \frac{\upsilon}{1-\upsilon x}\Big)$$

Usando (36) tenemos que

$$\sum_{k=0}^{\infty} \frac{(\sigma^{k+1} - \upsilon^{k+1})}{\sigma - \upsilon} x^k = \sum_{k=0}^{\infty} \varphi_{-(k+1)} x^k$$

por tanto si hacemos $k + 1 = k'$, obtenemos

$$\varphi_{-k'} = \frac{(\sigma^{k'} - \upsilon^{k'})}{\sigma - \upsilon} = \frac{(\sigma^{k'} - \upsilon^{k'})}{\sqrt{5}}$$

para todo $k' > 0$ ■

Teorema 20. *Sea* $k, n \in \mathbb{N}$

$$\sum_{k=0}^{n} (-1)^k \varphi_k = (-1)^{n+1} \varphi_n + (-1)^n \varphi_{n+1} - 1 \tag{42}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \frac{1-\sqrt{5}}{2}$, podemos usar (30)

$$\sum_{k=0}^{n} (-1)^k \varphi_k x^k = \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1}$$

y fijarnos en el límite cuando x tiende a 1

$$\begin{aligned}
\lim_{x\to 1} \Big(\sum_{k=0}^{n} (-1)^k \varphi_k x^k\Big) &= \lim_{x\to 1} \Big(\frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1}\Big) \\
\sum_{k=0}^{n} (-1)^k \varphi_k &= \frac{(-1)^n \varphi_n + (-1)^{n+1} \varphi_{n+1} + 1}{-1} \\
&= (-1)^{n+1} \varphi_n + (-1)^n \varphi_{n+1} - 1
\end{aligned}$$

■

Teorema 21. *Sea* $n \in \mathbb{N}$

$$\sum_{k=0}^{n} \varphi_{-k} = (-1)^n \varphi_n + (-1)^{n+1} \varphi_{n+1} + 1 \tag{43}$$

Demostración. Sea $n \in \mathbb{N}$, podemos usar (42)

$$\sum_{k=0}^{n} (-1)^k \varphi_k = (-1)^{n+1} \varphi_n + (-1)^n \varphi_{n+1} - 1$$

y multiplicar por (-1)

$$\sum_{k=0}^{n} \varphi_{-k} = \sum_{k=0}^{n} (-1)^{k+1} \varphi_k = (-1)^n \varphi_n + (-1)^{n+1} \varphi_{n+1} + 1$$

■

Corolario 5 (Fórmula de Medina)**.** *Sea* $n \in \mathbb{N}$

$$(-1)^{n-1} \varphi_{n-1} = (-1)^n \varphi_n + (-1)^{n+1} \varphi_{n+1} \tag{44}$$

Demostración. Igualamos (8) con (43) ■

Teorema 22. *Sea* $n \in \mathbb{Z}$, $\psi = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$, $\alpha = \frac{n}{|n|}$

$$\varphi_n = \alpha \Big(\frac{(\alpha\psi)^{|n|} - (\alpha\tau)^{|n|}}{\sqrt{5}} \Big) \tag{45}$$

Demostración. Sea $n \in \mathbb{Z}$
Si $n \geqslant 0$, podemos usar (28)

$$\varphi_n = \frac{(\psi)^n - (\tau)^n}{\sqrt{5}}$$

y tenemos que $\alpha = 1$ y $|n| = n$, por tanto

$$\varphi_n = \alpha \Big(\frac{(\alpha\psi)^{|n|} - (\alpha\tau)^{|n|}}{\sqrt{5}} \Big)$$

para todo $n \geqslant 0$
Si $n < 0$, podemos usar (40)

$$\varphi_n = \frac{(-\tau)^{-n} - (-\psi)^{-n}}{\sqrt{5}}$$
$$\varphi_n = (-1) \frac{(-\psi)^{-n} - (-\tau)^{-n}}{\sqrt{5}}$$

y tenemos que $\alpha = -1$ y $|n| = -n$, por tanto

$$\varphi_n = \alpha \Big(\frac{(\alpha\psi)^{|n|} - (\alpha\tau)^{|n|}}{\sqrt{5}} \Big)$$

para todo $n < 0$. ■

2.2 Suma de los φ_{2k}

Ahora vamos a generar un proceso para la construcción de más series, sumando y restando nuestra serie original (14) con la serie alternante (32). De tal manera que sumando obtenemos (46) y (47), y restando obtenemos (49) y (50).

Teorema 23. *Sea $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\text{Si } n \text{ es par, } \sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} = \frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2} x^n + 1)}{x^4 - 3x^2 + 1} \tag{46}$$

$$\text{Y si } n \text{ es impar, } \sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k} x^{2k} = \frac{x^2(\varphi_{n-1} x^{n+1} - \varphi_{n+1} x^{n-1} + 1)}{x^4 - 3x^2 + 1} \tag{47}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Si n es par, podemos usar (14)

$$\sum_{k=0}^{n} \varphi_k x^k = \frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2 + x - 1}$$

y sumar con (30)

$$\sum_{k=0}^{n} (-1)^k \varphi_k x^k = \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1}$$

obtenemos

$$\sum_{k=0}^{n} \varphi_k x^k + \sum_{k=0}^{n} (-1)^k \varphi_k x^k = \frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2 + x - 1} + \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1}$$

ya que n es par del lado izquierdo tenemos

$$\sum_{k=0}^{n} \varphi_k x^k + \sum_{k=0}^{n} (-1)^k \varphi_k x^k = \sum_{k=0}^{n} (1 + (-1)^k) \varphi_k x^k = 2 \sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k}$$

entonces

$$2 \sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} = \frac{(\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x)(x^2 - x - 1)}{(x^2 + x - 1)(x^2 - x - 1)} + \frac{((-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x)(x^2 + x - 1)}{(x^2 + x - 1)(x^2 - x - 1)}$$

$$
\begin{aligned}
2\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} = (&\varphi_n x^{n+4} + \varphi_{n+1} x^{n+3} - x^3 \\
&-\varphi_n x^{n+3} - \varphi_{n+1} x^{n+2} + x^2 \\
&-\varphi_n x^{n+2} - \varphi_{n+1} x^{n+1} + x \\
&+(-1)^n \varphi_n x^{n+4} + (-1)^{n+1} \varphi_{n+1} x^{n+3} + x^3 \\
&+(-1)^n \varphi_n x^{n+3} + (-1)^{n+1} \varphi_{n+1} x^{n+2} + x^2 \\
&+(-1)^{n+1} \varphi_n x^{n+2} + (-1)^n \varphi_{n+1} x^{n+1} - x) \frac{1}{(x^2+x-1)(x^2-x-1)}
\end{aligned}
$$

$$
\begin{aligned}
2\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} = (&\varphi_n x^{n+4} + \varphi_{n-1} x^{n+3} - \varphi_{n+2} x^{n+2} - \varphi_{n+1} x^{n+1} + 2x^2 \\
&+(-1)^n \varphi_n x^{n+4} + (-1)^{n+1} \varphi_{n-1} x^{n+3} + (-1)^{n+1} \varphi_{n+2} x^{n+2} \\
&+(-1)^n \varphi_{n+1} x^{n+1}) \frac{1}{(x^2+x-1)(x^2-x-1)}
\end{aligned}
$$

$$
\begin{aligned}
2\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} = ([&1+(-1)^{n+1}](-\varphi_n x^{n+4} + \varphi_{n-1} x^{n+3} + \varphi_{n+2} x^{n+2} - \varphi_{n+1} x^{n+1}) \\
&+2\varphi_n x^{n+4} - 2\varphi_{n+2} x^{n+2} + 2x^2) \frac{1}{(x^2+x-1)(x^2-x-1)}
\end{aligned}
$$

ya que n es par, $[1+(-1)^{n+1}] = 0$, y obtenemos

$$
\begin{aligned}
2\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} &= \frac{2\varphi_n x^{n+4} - 2\varphi_{n+2} x^{n+2} + 2x^2}{(x^2+x-1)(x^2-x-1)} \\
\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} &= \frac{\varphi_n x^{n+4} - \varphi_{n+2} x^{n+2} + x^2}{(x^2+x-1)(x^2-x-1)} \\
\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} &= \frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2} x^n + 1)}{(x^2+x-1)(x^2-x-1)} \\
\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} &= \frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2} x^n + 1)}{x^4 - 3x^2 + 1}
\end{aligned}
$$

Y si n es impar, podemos sumar (14) y (30) obtenemos

$$
\begin{aligned}
\sum_{k=0}^{n} \varphi_k x^k + \sum_{k=0}^{n} (-1)^k \varphi_k x^k = &\frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2+x-1} \\
&+ \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2-x-1}
\end{aligned}
$$

ya que n es impar del lado izquierdo tenemos

$$
\sum_{k=0}^{n} \varphi_k x^k + \sum_{k=0}^{n} (-1)^k \varphi_k x^k = \sum_{k=0}^{n} (1+(-1)^k) \varphi_k x^k = 2\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k} x^{2k}
$$

entonces

$$2\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} = \frac{(\varphi_n x^{n+2} + \varphi_{n+1}x^{n+1} - x)(x^2 - x - 1)}{(x^2 + x - 1)(x^2 - x - 1)} + \frac{((-1)^n\varphi_n x^{n+2} + (-1)^{n+1}\varphi_{n+1}x^{n+1} + x)(x^2 + x - 1)}{(x^2 + x - 1)(x^2 - x - 1)}$$

$$\begin{aligned}2\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} = (&\varphi_n x^{n+4} + \varphi_{n+1}x^{n+3} - x^3 \\ &-\varphi_n x^{n+3} - \varphi_{n+1}x^{n+2} + x^2 \\ &-\varphi_n x^{n+2} - \varphi_{n+1}x^{n+1} + x \\ &+(-1)^n\varphi_n x^{n+4} + (-1)^{n+1}\varphi_{n+1}x^{n+3} + x^3 \\ &+(-1)^n\varphi_n x^{n+3} + (-1)^{n+1}\varphi_{n+1}x^{n+2} + x^2 \\ &+(-1)^{n+1}\varphi_n x^{n+2} + (-1)^n\varphi_{n+1}x^{n+1} - x)\frac{1}{(x^2 + x - 1)(x^2 - x - 1)}\end{aligned}$$

$$\begin{aligned}2\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} = (&\varphi_n x^{n+4} + \varphi_{n-1}x^{n+3} - \varphi_{n+2}x^{n+2} - \varphi_{n+1}x^{n+1} + 2x^2 \\ &+(-1)^n\varphi_n x^{n+4} + (-1)^{n+1}\varphi_{n-1}x^{n+3} + (-1)^{n+1}\varphi_{n+2}x^{n+2} \\ &+(-1)^n\varphi_{n+1}x^{n+1})\frac{1}{(x^2 + x - 1)(x^2 - x - 1)}\end{aligned}$$

$$\begin{aligned}2\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} = (&[1 + (-1)^n](\varphi_n x^{n+4} - \varphi_{n-1}x^{n+3} - \varphi_{n+2}x^{n+2} + \varphi_{n+1}x^{n+1}) \\ &+2\varphi_{n-1}x^{n+3} - 2\varphi_{n+1}x^{n+1} + 2x^2)\frac{1}{(x^2 + x - 1)(x^2 - x - 1)}\end{aligned}$$

ya que n es impar, $[1 + (-1)^n] = 0$, y obtenemos

$$2\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} = \frac{2\varphi_{n-1}x^{n+3} - 2\varphi_{n+1}x^{n+1} + 2x^2}{(x^2 + x - 1)(x^2 - x - 1)}$$

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} = \frac{\varphi_{n-1}x^{n+3} - \varphi_{n+1}x^{n+1} + x^2}{(x^2 + x - 1)(x^2 - x - 1)}$$

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1)}{(x^2 + x - 1)(x^2 - x - 1)}$$

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1)}{x^4 - 3x^2 + 1}$$

■

Aquí tenemos una fórmula para expresar el caso par y el caso impar, utilizando el piso de n sobre dos.

Corolario 6. *Sea* $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \varphi_{2k}x^{2k} = \frac{x^2(\frac{[1+(-1)^n]}{2}(\varphi_n x^{n+2} - \varphi_{n+2}x^n) + \frac{[1+(-1)^{n+1}]}{2}(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1}) + 1)}{x^4 - 3x^2 + 1} \tag{48}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Si n es par, podemos usar (46)

$$\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} = \frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2} x^n + 1)}{x^4 - 3x^2 + 1}$$

ya que n es par tenemos que $\frac{n}{2} = \lfloor \frac{n}{2} \rfloor$, $\frac{[1+(-1)^n]}{2} = 1$, y $\frac{[1+(-1)^{n+1}]}{2} = 0$, con lo cual obtenemos

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \varphi_{2k} x^{2k} = \frac{x^2(\frac{[1+(-1)^n]}{2}(\varphi_n x^{n+2} - \varphi_{n+2} x^n) + 1)}{x^4 - 3x^2 + 1}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \varphi_{2k} x^{2k} = \frac{x^2(\frac{[1+(-1)^n]}{2}(\varphi_n x^{n+2} - \varphi_{n+2} x^n) + 0 + 1)}{x^4 - 3x^2 + 1}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \varphi_{2k} x^{2k} = \frac{x^2(\frac{[1+(-1)^n]}{2}(\varphi_n x^{n+2} - \varphi_{n+2} x^n) + \frac{[1+(-1)^{n+1}]}{2}(\varphi_{n-1} x^{n+1} - \varphi_{n+1} x^{n-1}) + 1)}{x^4 - 3x^2 + 1}$$

Y si n es impar, podemos usar (47)

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k} x^{2k} = \frac{x^2(\varphi_{n-1} x^{n+1} - \varphi_{n+1} x^{n-1} + 1)}{x^4 - 3x^2 + 1}$$

ya que n es impar tenemos que $\frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$, $\frac{[1+(-1)^n]}{2} = 0$, y $\frac{[1+(-1)^{n+1}]}{2} = 1$, con lo cual obtenemos

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \varphi_{2k} x^{2k} = \frac{x^2(\frac{[1+(-1)^{n+1}]}{2}(\varphi_{n-1} x^{n+1} - \varphi_{n+1} x^{n-1}) + 1)}{x^4 - 3x^2 + 1}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \varphi_{2k} x^{2k} = \frac{x^2(\frac{[1+(-1)^{n+1}]}{2}(\varphi_{n-1} x^{n+1} - \varphi_{n+1} x^{n-1}) + 0 + 1)}{x^4 - 3x^2 + 1}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \varphi_{2k} x^{2k} = \frac{x^2(\frac{[1+(-1)^{n+1}]}{2}(\varphi_{n-1} x^{n+1} - \varphi_{n+1} x^{n-1}) + \frac{[1+(-1)^n]}{2}(\varphi_n x^{n+2} - \varphi_{n+2} x^n) + 1)}{x^4 - 3x^2 + 1}$$

∎

Teorema 24. *Sea $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\textit{Si } n \textit{ es impar, } \sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} x^{2k+1} = \frac{x(\varphi_n x^{n+3} - \varphi_{n+2} x^{n+1} - x^2 + 1)}{x^4 - 3x^2 + 1} \quad (49)$$

$$\textit{Y si } n \textit{ es par, } \sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} x^{2k+1} = \frac{x(\varphi_{n-1} x^{n+2} - \varphi_{n+1} x^n - x^2 + 1)}{x^4 - 3x^2 + 1} \quad (50)$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Si n es impar, podemos usar (14)

$$\sum_{k=0}^{n} \varphi_k x^k = \frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2 + x - 1}$$

y restar con (30)

$$\sum_{k=0}^{n}(-1)^k\varphi_k x^k = \frac{(-1)^n\varphi_n x^{n+2} + (-1)^{n+1}\varphi_{n+1}x^{n+1} + x}{x^2 - x - 1}$$

obtenemos

$$\sum_{k=0}^{n}\varphi_k x^k - \sum_{k=0}^{n}(-1)^k\varphi_k x^k = \frac{\varphi_n x^{n+2} + \varphi_{n+1}x^{n+1} - x}{x^2 + x - 1} - \frac{(-1)^n\varphi_n x^{n+2} + (-1)^{n+1}\varphi_{n+1}x^{n+1} + x}{x^2 - x - 1}$$

ya que n es impar del lado izquierdo tenemos

$$\sum_{k=0}^{n}\varphi_k x^k - \sum_{k=0}^{n}(-1)^k\varphi_k x^k = \sum_{k=0}^{n}(1 + (-1)^{k+1})\varphi_k x^k = 2\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1}$$

entonces

$$2\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1} = \frac{(\varphi_n x^{n+2} + \varphi_{n+1}x^{n+1} - x)(x^2 - x - 1)}{(x^2 + x - 1)(x^2 - x - 1)} - \frac{((-1)^n\varphi_n x^{n+2} + (-1)^{n+1}\varphi_{n+1}x^{n+1} + x)(x^2 + x - 1)}{(x^2 + x - 1)(x^2 - x - 1)}$$

$$\begin{aligned}
2\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1} = (&\varphi_n x^{n+4} + \varphi_{n+1}x^{n+3} - x^3\\
&-\varphi_n x^{n+3} - \varphi_{n+1}x^{n+2} + x^2\\
&-\varphi_n x^{n+2} - \varphi_{n+1}x^{n+1} + x\\
&+(-1)^{n+1}\varphi_n x^{n+4} + (-1)^n\varphi_{n+1}x^{n+3} - x^3\\
&+(-1)^{n+1}\varphi_n x^{n+3} + (-1)^n\varphi_{n+1}x^{n+2} - x^2\\
&+(-1)^n\varphi_n x^{n+2} + (-1)^{n+1}\varphi_{n+1}x^{n+1} + x)\frac{1}{(x^2 + x - 1)(x^2 - x - 1)}\\
2\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1} = (&\varphi_n x^{n+4} + \varphi_{n-1}x^{n+3} - \varphi_{n+2}x^{n+2} - \varphi_{n+1}x^{n+1} - 2x^3 + 2x\\
&+(-1)^{n+1}\varphi_n x^{n+4} + (-1)^n\varphi_{n-1}x^{n+3} + (-1)^n\varphi_{n+2}x^{n+2}\\
&+(-1)^{n+1}\varphi_{n+1}x^{n+1})\frac{1}{(x^2 + x - 1)(x^2 - x - 1)}\\
2\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1} = (&[1 + (-1)^n](-\varphi_n x^{n+4} + \varphi_{n-1}x^{n+3} + \varphi_{n+2}x^{n+2} - \varphi_{n+1}x^{n+1})\\
&+2\varphi_n x^{n+4} - 2\varphi_{n+2}x^{n+2} - 2x^3 + 2x)\frac{1}{(x^2 + x - 1)(x^2 - x - 1)}
\end{aligned}$$

ya que n es impar, $[1+(-1)^n]=0$, y obtenemos

$$2\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1}=\frac{2\varphi_n x^{n+4}-2\varphi_{n+2}x^{n+2}-2x^3+2x}{(x^2+x-1)(x^2-x-1)}$$
$$\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1}=\frac{\varphi_n x^{n+4}-\varphi_{n+2}x^{n+2}-x^3+x}{(x^2+x-1)(x^2-x-1)}$$
$$\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1}=\frac{x(\varphi_n x^{n+3}-\varphi_{n+2}x^{n+1}-x^2+1)}{(x^2+x-1)(x^2-x-1)}$$
$$\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k+1}x^{2k+1}=\frac{x(\varphi_n x^{n+3}-\varphi_{n+2}x^{n+1}-x^2+1)}{x^4-3x^2+1}$$

Y si n es par, podemos restar (14) y (30) obtenemos

$$\sum_{k=0}^{n}\varphi_k x^k-\sum_{k=0}^{n}(-1)^k\varphi_k x^k=\frac{\varphi_n x^{n+2}+\varphi_{n+1}x^{n+1}-x}{x^2+x-1}-\frac{(-1)^n\varphi_n x^{n+2}+(-1)^{n+1}\varphi_{n+1}x^{n+1}+x}{x^2-x-1}$$

ya que n es par del lado izquierdo tenemos

$$\sum_{k=0}^{n}\varphi_k x^k-\sum_{k=0}^{n}(-1)^k\varphi_k x^k=\sum_{k=0}^{n}(1+(-1)^{k+1})\varphi_k x^k=2\sum_{k=0}^{\frac{n-2}{2}}\varphi_{2k+1}x^{2k+1}$$

entonces

$$2\sum_{k=0}^{\frac{n-2}{2}}\varphi_{2k+1}x^{2k+1}=\frac{(\varphi_n x^{n+2}+\varphi_{n+1}x^{n+1}-x)(x^2-x-1)}{(x^2+x-1)(x^2-x-1)}-\frac{((-1)^n\varphi_n x^{n+2}+(-1)^{n+1}\varphi_{n+1}x^{n+1}+x)(x^2+x-1)}{(x^2+x-1)(x^2-x-1)}$$

$$\begin{aligned}2\sum_{k=0}^{\frac{n-2}{2}}\varphi_{2k+1}x^{2k+1}=(&\varphi_n x^{n+4}+\varphi_{n+1}x^{n+3}-x^3\\&-\varphi_n x^{n+3}-\varphi_{n+1}x^{n+2}+x^2\\&-\varphi_n x^{n+2}-\varphi_{n+1}x^{n+1}+x\\&+(-1)^{n+1}\varphi_n x^{n+4}+(-1)^n\varphi_{n+1}x^{n+3}-x^3\\&+(-1)^{n+1}\varphi_n x^{n+3}+(-1)^n\varphi_{n+1}x^{n+2}-x^2\\&+(-1)^n\varphi_n x^{n+2}+(-1)^{n+1}\varphi_{n+1}x^{n+1}+x)\frac{1}{(x^2+x-1)(x^2-x-1)}\end{aligned}$$

$$2\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1}x^{2k+1} = (\varphi_n x^{n+4} + \varphi_{n-1}x^{n+3} - \varphi_{n+2}x^{n+2} - \varphi_{n+1}x^{n+1} - 2x^3 + 2x$$
$$+(-1)^{n+1}\varphi_n x^{n+4} + (-1)^n\varphi_{n-1}x^{n+3} + (-1)^n\varphi_{n+2}x^{n+2}$$
$$+(-1)^{n+1}\varphi_{n+1}x^{n+1})\frac{1}{(x^2+x-1)(x^2-x-1)}$$

$$2\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1}x^{2k+1} = ([1+(-1)^{n+1}](\varphi_n x^{n+4} - \varphi_{n-1}x^{n+3} - \varphi_{n+2}x^{n+2} + \varphi_{n+1}x^{n+1})$$
$$+2\varphi_{n-1}x^{n+3} - 2\varphi_{n+1}x^{n+1} - 2x^3 + 2x)\frac{1}{(x^2+x-1)(x^2-x-1)}$$

ya que n es par, $[1+(-1)^{n+1}] = 0$, y obtenemos

$$2\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1}x^{2k+1} = \frac{2\varphi_{n-1}x^{n+3} - 2\varphi_{n+1}x^{n+1} - 2x^3 + 2x}{(x^2+x-1)(x^2-x-1)}$$
$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1}x^{2k+1} = \frac{\varphi_{n-1}x^{n+3} - \varphi_{n+1}x^{n+1} - x^3 + x}{(x^2+x-1)(x^2-x-1)}$$
$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1}x^{2k+1} = \frac{x(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n} - x^2 + 1)}{(x^2+x-1)(x^2-x-1)}$$
$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1}x^{2k+1} = \frac{x(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n} - x^2 + 1)}{x^4 - 3x^2 + 1}$$

■

Corolario 7. *Sea $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \varphi_{2k+1}x^{2k+1} = \frac{x(\frac{[1+(-1)^{n+1}]}{2}(\varphi_n x^{n+3} - \varphi_{n+2}x^{n+1}) + \frac{[1+(-1)^n]}{2}(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^n) - x^2 + 1)}{x^4 - 3x^2 + 1} \tag{51}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Si n es impar, podemos usar (49)

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1}x^{2k+1} = \frac{x(\varphi_n x^{n+3} - \varphi_{n+2}x^{n+1} - x^2 + 1)}{x^4 - 3x^2 + 1}$$

ya que n es impar tenemos que $\frac{n-1}{2} = \lfloor\frac{n-1}{2}\rfloor$, $\frac{[1+(-1)^n]}{2} = 0$, y $\frac{[1+(-1)^{n+1}]}{2} = 1$, con lo cual obtenemos

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \varphi_{2k+1}x^{2k+1} = \frac{x(\frac{[1+(-1)^{n+1}]}{2}(\varphi_n x^{n+3} - \varphi_{n+2}x^{n+1}) - x^2 + 1)}{x^4 - 3x^2 + 1}$$
$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \varphi_{2k+1}x^{2k+1} = \frac{x(\frac{[1+(-1)^{n+1}]}{2}(\varphi_n x^{n+3} - \varphi_{n+2}x^{n+1}) + 0 - x^2 + 1)}{x^4 - 3x^2 + 1}$$
$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \varphi_{2k+1}x^{2k+1} = \frac{x(\frac{[1+(-1)^{n+1}]}{2}(\varphi_n x^{n+3} - \varphi_{n+2}x^{n+1}) + \frac{[1+(-1)^n]}{2}(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^n) - x^2 + 1)}{x^4 - 3x^2 + 1}$$

Y si n es par, podemos usar (50)

$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1}x^{2k+1} = \frac{x(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n} - x^2 + 1)}{x^4 - 3x^2 + 1}$$

ya que n es par tenemos que $\frac{n-2}{2} = \left\lfloor \frac{n-1}{2} \right\rfloor$, $\frac{[1+(-1)^n]}{2} = 1$, y $\frac{[1+(-1)^{n+1}]}{2} = 0$, con lo cual obtenemos

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \varphi_{2k+1}x^{2k+1} = \frac{x(\frac{[1+(-1)^n]}{2}(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n}) - x^2 + 1)}{x^4 - 3x^2 + 1}$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \varphi_{2k+1}x^{2k+1} = \frac{x(\frac{[1+(-1)^n]}{2}(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n}) + 0 - x^2 + 1)}{x^4 - 3x^2 + 1}$$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \varphi_{2k+1}x^{2k+1} = \frac{x(\frac{[1+(-1)^n]}{2}(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n}) + \frac{[1+(-1)^{n+1}]}{2}(\varphi_{n}x^{n+3} - \varphi_{n+2}x^{n+1}) - x^2 + 1)}{x^4 - 3x^2 + 1}$$

■

Teorema 25. *Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$*

$$\lim_{n\to\infty} \Big(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1\Big) = 1 \tag{52}$$

$$\lim_{n\to\infty} \Big(\varphi_{n}x^{n+2} - \varphi_{n+2}x^{n} + 1\Big) = 1 \tag{53}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$

$$\big|(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1) - (1)\big| = \big|\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1}\big|$$

Ya que $0 < x < 1$, podemos escribir $x = \frac{1}{y}$ para alguna $y \neq 0$

$$\big|\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1}\big| = \left|\varphi_{n-1}\frac{1}{y^{n+1}} - \varphi_{n+1}\frac{1}{y^{n-1}}\right|$$

Sea $\epsilon \in \mathbb{R}$, $\epsilon > 0$, $a = \varphi_{n+1}, b = y$ entonces queremos encontrar a partir de cual δ se cumple lo siguiente

$$\frac{a}{b^{\delta-1}} < \frac{\epsilon}{2}$$

Despejando δ tenemos que

$$2a * \epsilon^{-1} < b^{\delta-1}$$
$$\log(2a * \epsilon^{-1}) < \log(b^{\delta-1})$$
$$\log(2a) - \log(\epsilon) < (\delta - 1)\log(b)$$
$$\frac{\log(2a) - \log(\epsilon)}{\log(b)} + 1 < \delta$$

con esta δ y sabiendo que $y > 0$ tenemos que

$$-\frac{\epsilon}{2} < \frac{\varphi_{n-1}}{y^{\delta+1}} \leqslant \frac{\varphi_{n}}{y^{\delta+1}} \leqslant \frac{\varphi_{n+1}}{y^{\delta+1}} < \frac{\varphi_{n+1}}{y^{\delta}} < \frac{\varphi_{n+1}}{y^{\delta-1}} < \frac{\epsilon}{2}$$

y además tenemos

$$-\frac{\epsilon}{2} < -\frac{\varphi_{n-1}}{y^{\delta+1}} \leqslant -\frac{\varphi_{n}}{y^{\delta+1}} \leqslant -\frac{\varphi_{n+1}}{y^{\delta+1}} < -\frac{\varphi_{n+1}}{y^{\delta}} < -\frac{\varphi_{n+1}}{y^{\delta-1}} < \frac{\epsilon}{2}$$

con lo cual resulta

$$-\epsilon < -\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1} < \epsilon$$
$$-\epsilon < \varphi_{n+1}x^{n-1} - \varphi_{n-1}x^{n+1} + 1 - 1 < \epsilon$$
$$\left|(\varphi_{n+1}x^{n-1} - \varphi_{n-1}x^{n+1} + 1) - (1)\right| < \epsilon$$

para todo $n > \delta$, entonces

$$\lim_{n\to\infty}\left(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1\right) = 1$$

y haciendo $n = n' + 1$, tenemos

$$\lim_{n\to\infty}\left(\varphi_{n'}x^{n'+2} - \varphi_{n'+2}x^{n'} + 1\right) = 1$$

■

Teorema 26. *Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$*

$$\lim_{n\to\infty}\left(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n} - x^2 + 1\right) = 1 - x^2 \tag{54}$$

$$\lim_{n\to\infty}\left(\varphi_{n}x^{n+3} - \varphi_{n+2}x^{n+1} - x^2 + 1\right) = 1 - x^2 \tag{55}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 0$

$$\left|(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n} + -x^2 + 1) - (1 - x^2)\right| = \left|\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n}\right|$$

Ya que $0 < x < 1$, podemos escribir $x = \frac{1}{y}$ para alguna $y \neq 0$

$$\left|\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^{n}\right| = \left|\varphi_{n-1}\frac{1}{y^{n+2}} - \varphi_{n+1}\frac{1}{y^{n}}\right|$$

Sea $\epsilon \in \mathbb{R}$, $\epsilon > 0$, $a = \varphi_{n+1}, b = y$ entonces queremos encontrar a partir de cual δ se cumple lo siguiente

$$\frac{a}{b^\delta} < \frac{\epsilon}{2}$$

Despejando δ tenemos que

$$2a * \epsilon^{-1} < b^\delta$$
$$\log(2a * \epsilon^{-1}) < \log(b^\delta)$$
$$\log(2a) - \log(\epsilon) < (\delta)\log(b)$$
$$\frac{\log(2a) - \log(\epsilon)}{\log(b)} < \delta$$

con esta δ y sabiendo que $y > 0$ tenemos que

$$-\frac{\epsilon}{2} < \frac{\varphi_{n-1}}{y^{\delta+2}} \leqslant \frac{\varphi_{n}}{y^{\delta+2}} \leqslant \frac{\varphi_{n+1}}{y^{\delta+2}} < \frac{\varphi_{n+1}}{y^{\delta+1}} < \frac{\varphi_{n+1}}{y^{\delta}} < \frac{\epsilon}{2}$$

y además tenemos

$$-\frac{\epsilon}{2} < -\frac{\varphi_{n-1}}{y^{\delta+2}} \leqslant -\frac{\varphi_{n}}{y^{\delta+2}} \leqslant -\frac{\varphi_{n+1}}{y^{\delta+2}} < -\frac{\varphi_{n+1}}{y^{\delta+1}} < -\frac{\varphi_{n+1}}{y^{\delta}} < \frac{\epsilon}{2}$$

con lo cual resulta

$$\begin{aligned} -\epsilon < \varphi_{n-1}x^{n+2} - \varphi_{n+1}x^n < \epsilon \\ -\epsilon < (\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^n - x^2 + 1) - (1 - x^2) < \epsilon \\ \left|(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^n - x^2 + 1) - (1 - x^2)\right| < \epsilon \end{aligned}$$

para todo $n > \delta$, entonces

$$\lim_{n\to\infty}\Big(\varphi_{n-1}x^{n+2} - \varphi_{n+1}x^n - x^2 + 1\Big) = 1 - x^2$$

y haciendo $n = n' + 1$, tenemos

$$\lim_{n\to\infty}\Big(\varphi_{n'}x^{n'+3} - \varphi_{n'+2}x^{n'+1} - x^2 + 1\Big) = 1 - x^2$$

■

Corolario 8. *Sea $k \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\sum_{k=0}^{\infty}\varphi_{2k}x^{2k} = \frac{x^2}{x^4 - 3x^2 + 1} \tag{56}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Si n es par entonces podemos usar (46)

$$\sum_{k=0}^{\frac{n}{2}}\varphi_{2k}x^{2k} = \frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2}x^n + 1)}{x^4 - 3x^2 + 1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned} \sum_{k=0}^{\infty}\varphi_{2k}x^{2k} &= \lim_{n\to\infty}\Big(\frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2}x^n + 1)}{x^4 - 3x^2 + 1}\Big) \\ &= \lim_{n\to\infty}\Big(\varphi_n x^{n+2} - \varphi_{n+2}x^n + 1\Big)\frac{x^2}{x^4 - 3x^2 + 1} \\ \text{y aplicando (53)} & \\ &= \frac{x^2}{x^4 - 3x^2 + 1} \end{aligned}$$

Y si n es impar entonces podemos usar (47)

$$\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k}x^{2k} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1)}{x^4 - 3x^2 + 1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned} \sum_{k=0}^{\infty}\varphi_{2k}x^{2k} &= \lim_{n\to\infty}\Big(\frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1)}{x^4 - 3x^2 + 1}\Big) \\ &= \lim_{n\to\infty}\Big(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1\Big)\frac{x^2}{x^4 - 3x^2 + 1} \\ \text{y aplicando (52)} & \\ &= \frac{x^2}{x^4 - 3x^2 + 1} \end{aligned}$$

■

Corolario 9. *Sea $k \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\sum_{k=0}^{\infty} \varphi_{2k+1} x^{2k+1} = \frac{x(1-x^2)}{x^4 - 3x^2 + 1} \tag{57}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Si n es impar entonces podemos usar (49)

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} x^{2k+1} = \frac{x(\varphi_n x^{n+3} - \varphi_{n+2} x^{n+1} - x^2 + 1)}{x^4 - 3x^2 + 1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned}\sum_{k=0}^{\infty} \varphi_{2k+1} x^{2k+1} &= \lim_{n\to\infty} \left(\frac{x(\varphi_n x^{n+3} - \varphi_{n+2} x^{n+1} - x^2 + 1)}{x^4 - 3x^2 + 1} \right) \\ &= \lim_{n\to\infty} \left(\varphi_n x^{n+3} - \varphi_{n+2} x^{n+1} - x^2 + 1 \right) \frac{x}{x^4 - 3x^2 + 1} \\ &\text{y aplicando (55)} \\ &= \frac{x(1-x^2)}{x^4 - 3x^2 + 1}\end{aligned}$$

Y si n es par entonces podemos usar (50)

$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} x^{2k+1} = \frac{x(\varphi_{n-1} x^{n+2} - \varphi_{n+1} x^{n} - x^2 + 1)}{x^4 - 3x^2 + 1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned}\sum_{k=0}^{\infty} \varphi_{2k+1} x^{2k+1} &= \lim_{n\to\infty} \left(\frac{x(\varphi_{n-1} x^{n+2} - \varphi_{n+1} x^{n} - x^2 + 1)}{x^4 - 3x^2 + 1} \right) \\ &= \lim_{n\to\infty} \left(\varphi_{n-1} x^{n+2} - \varphi_{n+1} x^{n} - x^2 + 1 \right) \frac{x}{x^4 - 3x^2 + 1} \\ &\text{y aplicando (54)} \\ &= \frac{x(1-x^2)}{x^4 - 3x^2 + 1}\end{aligned}$$

■

Aquí tenemos que para obtener $(-1)^k$ tenemos que realizar la siguiente sustitución $x = ix$, donde $i = \sqrt{-1} \in \mathbb{C}$.

Teorema 27. *Sea $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $i = \sqrt{-1} \in \mathbb{C}$*

$$\text{Si } n \text{ es par,} \sum_{k=0}^{\frac{n}{2}} (-1)^k \varphi_{2k} x^{2k} = \frac{x^2(i^n(\varphi_n x^{n+2} + \varphi_{n+2} x^n) - 1)}{x^4 + 3x^2 + 1} \tag{58}$$

$$\text{Y si } n \text{ es impar,} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \varphi_{2k} x^{2k} = \frac{x^2(i^{n-1}(\varphi_{n-1} x^{n+1} + \varphi_{n+1} x^{n-1}) - 1)}{x^4 + 3x^2 + 1} \tag{59}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $i = \sqrt{-1} \in \mathbb{C}$, si n es par, podemos usar (46)

$$\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} = \frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2} x^n + 1)}{x^4 - 3x^2 + 1}$$

haciendo $x = ix$, y sabiendo que $i^2 = -1$, $i^4 = 1$, tenemos

$$\sum_{k=0}^{\frac{n}{2}} \varphi_{2k} i^{2k} x^{2k} = \frac{i^2x^2(\varphi_n i^{n+2}x^{n+2} - \varphi_{n+2} i^n x^n + 1)}{i^4x^4 - i^2 3x^2 + 1}$$

$$\sum_{k=0}^{\frac{n}{2}} \varphi_{2k}(-1)^k x^{2k} = \frac{-x^2(-\varphi_n i^n x^{n+2} - \varphi_{n+2} i^n x^n + 1)}{x^4 + 3x^2 + 1}$$

$$\sum_{k=0}^{\frac{n}{2}} (-1)^k \varphi_{2k} x^{2k} = \frac{x^2(i^n(\varphi_n x^{n+2} + \varphi_{n+2}x^n) - 1)}{x^4 + 3x^2 + 1}$$

Y si n es impar, podemos usar (47)

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k} x^{2k} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1)}{x^4 - 3x^2 + 1}$$

haciendo $x = ix$, y sabiendo que $i^2 = -1$, $i^4 = 1$, tenemos

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k} i^{2k} x^{2k} = \frac{i^2x^2(\varphi_{n-1} i^{n+1}x^{n+1} - \varphi_{n+1} i^{n-1} x^{n-1} + 1)}{i^4x^4 - i^2 3x^2 + 1}$$

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}(-1)^k x^{2k} = \frac{-x^2(-\varphi_{n-1} i^{n-1} x^{n+1} - \varphi_{n+1} i^{n-1} x^{n-1} + 1)}{x^4 + 3x^2 + 1}$$

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \varphi_{2k} x^{2k} = \frac{x^2(i^{n-1}(\varphi_{n-1} x^{n+1} + \varphi_{n+1}x^{n-1}) - 1)}{x^4 + 3x^2 + 1}$$

■

Teorema 28. *Sea $k, n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $i = \sqrt{-1} \in \mathbb{C}$*

$$\text{Si } n \text{ es impar, } \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \varphi_{2k+1} x^{2k+1} = \frac{x(i^{n-1}(\varphi_n x^{n+3} + \varphi_{n+2}x^{n+1}) + x^2 + 1)}{x^4 + 3x^2 + 1} \tag{60}$$

$$\text{Y si } n \text{ es par, } \sum_{k=0}^{\frac{n-2}{2}} (-1)^k \varphi_{2k+1} x^{2k+1} = \frac{x(i^{n-2}(\varphi_{n-1} x^{n+2} + \varphi_{n+1}x^{n}) + x^2 + 1)}{x^4 + 3x^2 + 1} \tag{61}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $i = \sqrt{-1} \in \mathbb{C}$
Si n es impar, podemos usar (49)

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} x^{2k+1} = \frac{x(\varphi_n x^{n+3} - \varphi_{n+2}x^{n+1} - x^2 + 1)}{x^4 - 3x^2 + 1}$$

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} x^{2k} = \frac{\varphi_n x^{n+3} - \varphi_{n+2}x^{n+1} - x^2 + 1}{x^4 - 3x^2 + 1}$$

haciendo $x = ix$, y sabiendo que $i^2 = -1$, $i^4 = 1$, tenemos

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} i^{2k} x^{2k} = \frac{\varphi_n i^{n+3} x^{n+3} - \varphi_{n+2} i^{n+1} x^{n+1} - i^2 x^2 + 1}{i^4 x^4 - i^2 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} (-1)^k x^{2k} = \frac{-\varphi_n i^{n+1} x^{n+3} - \varphi_{n+2} i^{n+1} x^{n+1} + x^2 + 1}{x^4 + 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} (-1)^k x^{2k} = \frac{\varphi_n i^{n-1} x^{n+3} + \varphi_{n+2} i^{n-1} x^{n+1} + x^2 + 1}{x^4 + 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} (-1)^k x^{2k} = \frac{i^{n-1}(\varphi_n x^{n+3} + \varphi_{n+2} x^{n+1}) + x^2 + 1}{x^4 + 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \varphi_{2k+1} x^{2k+1} = \frac{x(i^{n-1}(\varphi_n x^{n+3} + \varphi_{n+2} x^{n+1}) + x^2 + 1)}{x^4 + 3x^2 + 1}$$

Y si n es par, podemos usar (50)

$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} x^{2k+1} = \frac{x(\varphi_{n-1} x^{n+2} - \varphi_{n+1} x^n - x^2 + 1)}{x^4 - 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} x^{2k} = \frac{\varphi_{n-1} x^{n+2} - \varphi_{n+1} x^n - x^2 + 1}{x^4 - 3x^2 + 1}$$

haciendo $x = ix$, y sabiendo que $i^2 = -1$, $i^4 = 1$, tenemos

$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} i^{2k} x^{2k} = \frac{\varphi_{n-1} i^{n+2} x^{n+2} - \varphi_{n+1} i^n x^n - i^2 x^2 + 1}{i^4 x^4 - i^2 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} (-1)^k x^{2k} = \frac{-\varphi_{n-1} i^n x^{n+2} - \varphi_{n+1} i^n x^n + x^2 + 1}{x^4 + 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} (-1)^k x^{2k} = \frac{\varphi_{n-1} i^{n-2} x^{n+2} + \varphi_{n+1} i^{n-2} x^n + x^2 + 1}{x^4 + 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} (-1)^k x^{2k} = \frac{i^{n-2}(\varphi_{n-1} x^{n+2} + \varphi_{n+1} x^n) + x^2 + 1}{x^4 + 3x^2 + 1}$$
$$\sum_{k=0}^{\frac{n-2}{2}} (-1)^k \varphi_{2k+1} x^{2k+1} = \frac{x(i^{n-2}(\varphi_{n-1} x^{n+2} + \varphi_{n+1} x^n) + x^2 + 1)}{x^4 + 3x^2 + 1}$$

■

Teorema 29. *Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $i = \sqrt{-1} \in \mathbb{C}$*

$$\lim_{n\to\infty} \Big(i^{n-1}(\varphi_{n-1} x^{n+1} + \varphi_{n+1} x^{n-1}) - 1 \Big) = -1 \tag{62}$$

$$\lim_{n\to\infty} \Big(i^n(\varphi_n x^{n+2} + \varphi_{n+2} x^n) - 1 \Big) = -1 \tag{63}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $i = \sqrt{-1} \in \mathbb{C}$

$$\left|(i^{n-1}(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1}) - 1) - (-1)\right| = \left|i^{n-1}(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\right|$$

Ya que $0 < x < 1$, podemos escribir $x = \frac{1}{y}$ para alguna $y \neq 0$

$$\left|i^{n-1}(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\right| = \left|i^{n-1}(\varphi_{n-1}\frac{1}{y^{n+1}} + \varphi_{n+1}\frac{1}{y^{n-1}})\right|$$

Sea $\epsilon \in \mathbb{R}$, $\epsilon > 0$, $a = \varphi_{n+1}, b = y$ entonces queremos encontrar a partir de cual δ se cumple lo siguiente

$$\frac{a}{b^{\delta-1}} < \frac{\epsilon}{2}$$

Despejando δ tenemos que

$$\begin{aligned} 2a * \epsilon^{-1} &< b^{\delta-1} \\ \log(2a * \epsilon^{-1}) &< \log(b^{\delta-1}) \\ \log(2a) - \log(\epsilon) &< (\delta - 1)\log(b) \\ \frac{\log(2a) - \log(\epsilon)}{\log(b)} + 1 &< \delta \end{aligned}$$

con esta δ y sabiendo que $y > 0$ tenemos que

$$0 < \frac{\varphi_{n-1}}{y^{\delta+1}} \leqslant \frac{\varphi_n}{y^{\delta+1}} \leqslant \frac{\varphi_{n+1}}{y^{\delta+1}} < \frac{\varphi_{n+1}}{y^{\delta}} < \frac{\varphi_{n+1}}{y^{\delta-1}} < \frac{\epsilon}{2}$$

con lo cual resulta

$$\begin{aligned} 0 < \varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1} &< \epsilon \\ -\epsilon < \varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1} &< \epsilon \\ \left|\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1}\right| &< \epsilon \end{aligned}$$

para toda $n > \delta$ por tanto

$$\lim_{n\to\infty} \left(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1}\right) = 0$$

y si multiplicamos por i tenemos

$$\lim_{n\to\infty} \left(i(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\right) = 0$$

es decir que para todo ϵ_1 existe una δ_1 tal que para todo $n > \delta_1$ se cumple con lo siguiente

$$\begin{aligned} \left|i(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\right| &< \epsilon_1 \\ |i| \left|(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\right| &< \epsilon_1 \end{aligned}$$

de la misma manera si dividimos el límite por i obtenemos

$$\lim_{n\to\infty} \left(\frac{(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})}{i}\right) = 0$$

es decir que para todo ϵ_2 existe una δ_2 tal que para todo $n > \delta_2$ se cumple con lo siguiente

$$\begin{aligned} \left|\frac{(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})}{i}\right| &< \epsilon_2 \\ \left|\frac{1}{i}\right| \left|(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\right| &< \epsilon_2 \end{aligned}$$

y sabiendo que $\left|i^{n-1}\right| = \{\frac{1}{i}, 1, i\}$, sea $\epsilon_3 = max\{\epsilon, \epsilon_1, \epsilon_2\}$, y sea $\delta_3 = max\{\delta, \delta_1, \delta_2\}$ tenemos que

$$\left|i^{n-1}\right| \left|(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\right| < \epsilon_3$$
$$\left|i^{n-1}(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\right| < \epsilon_3$$

para toda $n > \delta_3$, por tanto

$$\lim_{n\to\infty} \Big(i^{n-1}(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1})\Big) = 0$$

con lo cual

$$\lim_{n\to\infty} \Big(i^{n-1}(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1}) - 1\Big) = -1$$

y haciendo $n = n + 1$, obtenemos

$$\lim_{n\to\infty} \Big(i^{n}(\varphi_{n}x^{n+2} + \varphi_{n+2}x^{n}) - 1\Big) = -1$$

■

Teorema 30. *Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $i = \sqrt{-1} \in \mathbb{C}$*

$$\lim_{n\to\infty} \Big(i^{n-2}(\varphi_{n-1}x^{n+2} + \varphi_{n+1}x^{n}) + x^2 + 1\Big) = 1 + x^2 \tag{64}$$

$$\lim_{n\to\infty} \Big(i^{n-1}(\varphi_{n}x^{n+3} + \varphi_{n+2}x^{n+1}) + x^2 + 1\Big) = 1 + x^2 \tag{65}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $i = \sqrt{-1} \in \mathbb{C}$

$$\left|(i^{n-2}(\varphi_{n-1}x^{n+2} + \varphi_{n+1}x^{n}) + x^2 + 1) - (1 + x^2)\right| = \left|i^{n-2}(\varphi_{n-1}x^{n+2} + \varphi_{n+1}x^{n})\right|$$

Ya que $0 < x < 1$, podemos escribir $x = \frac{1}{y}$ para alguna $y \neq 0$

$$\left|i^{n-2}(\varphi_{n-1}x^{n+2} + \varphi_{n+1}x^{n})\right| = \left|i^{n-2}(\varphi_{n-1}\frac{1}{y^{n+2}} + \varphi_{n+1}\frac{1}{y^{n}})\right|$$

Sea $\epsilon \in \mathbb{R}$, $\epsilon > 0$, $a = \varphi_{n+1}, b = y$ entonces queremos encontrar a partir de cual δ se cumple lo siguiente

$$\frac{a}{b^\delta} < \frac{\epsilon}{2}$$

Despejando δ tenemos que

$$2a * \epsilon^{-1} < b^\delta$$
$$\log(2a * \epsilon^{-1}) < \log(b^\delta)$$
$$\log(2a) - \log(\epsilon) < (\delta)\log(b)$$
$$\frac{\log(2a) - \log(\epsilon)}{\log(b)} < \delta$$

con esta δ y sabiendo que $y > 0$ tenemos que

$$0 < \frac{\varphi_{n-1}}{y^{\delta+2}} \leqslant \frac{\varphi_n}{y^{\delta+2}} \leqslant \frac{\varphi_{n+1}}{y^{\delta+2}} < \frac{\varphi_{n+1}}{y^{\delta+1}} < \frac{\varphi_{n+1}}{y^{\delta}} < \frac{\epsilon}{2}$$

con lo cual resulta

$$0 < \varphi_{n-1}x^{n+2} + \varphi_{n+1}x^{n} < \epsilon$$
$$-\epsilon < \varphi_{n-1}x^{n+2} + \varphi_{n+1}x^{n} < \epsilon$$
$$\left|\varphi_{n-1}x^{n+2} + \varphi_{n+1}x^{n}\right| < \epsilon$$

para toda $n > \delta$ por tanto

$$\lim_{n\to\infty}\Big(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n}\Big)=0$$

y si multiplicamos por i tenemos

$$\lim_{n\to\infty}\Big(i(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})\Big)=0$$

es decir que para todo ϵ_1 existe una δ_1 tal que para todo $n > \delta_1$ se cumple con lo siguiente

$$\begin{aligned}\left|i(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})\right| &< \epsilon_1\\ |i|\left|(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})\right| &< \epsilon_1\end{aligned}$$

de la misma manera si dividimos el límite por i obtenemos

$$\lim_{n\to\infty}\Big(\frac{(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})}{i}\Big)=0$$

es decir que para todo ϵ_2 existe una δ_2 tal que para todo $n > \delta_2$ se cumple con lo siguiente

$$\begin{aligned}\left|\frac{(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})}{i}\right| &< \epsilon_2\\ \left|\frac{1}{i}\right|\left|(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})\right| &< \epsilon_2\end{aligned}$$

y sabiendo que $\left|i^{n-2}\right| = \{1, \frac{1}{i}, i\}$, sea $\epsilon_3 = max\{\epsilon, \epsilon_1, \epsilon_2\}$, y sea $\delta_3 = max\{\delta, \delta_1, \delta_2\}$ tenemos que

$$\begin{aligned}\left|i^{n-2}\right|\left|(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})\right| &< \epsilon_3\\ \left|i^{n-2}(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})\right| &< \epsilon_3\end{aligned}$$

para toda $n > \delta_3$, por tanto

$$\lim_{n\to\infty}\Big(i^{n-2}(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})\Big)=0$$

con lo cual

$$\lim_{n\to\infty}\Big(i^{n-2}(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^{n})+x^2+1\Big)=1+x^2$$

y haciendo $n = n + 1$, obtenemos

$$\lim_{n\to\infty}\Big(i^{n-1}(\varphi_{n}x^{n+3}+\varphi_{n+2}x^{n+1})+x^2+1\Big)=1+x^2$$

■

Corolario 10. *Sea* $k \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$

$$\sum_{k=0}^{\infty}(-1)^k\varphi_{2k}x^{2k}=\frac{-x^2}{x^4+3x^2+1} \tag{66}$$

Demostración. Sea $n \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$
Si n es par, podemos usar (58)

$$\sum_{k=0}^{\frac{n}{2}}(-1)^k\varphi_{2k}x^{2k}=\frac{x^2(i^n(\varphi_n x^{n+2}+\varphi_{n+2}x^n)-1)}{x^4+3x^2+1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\sum_{k=0}^{\infty}(-1)^k\varphi_{2k}x^{2k} = \lim_{n\to\infty}\Big(\frac{x^2(i^n(\varphi_n x^{n+2}+\varphi_{n+2}x^n)-1)}{x^4+3x^2+1}\Big)$$
$$= \lim_{n\to\infty}\Big(i^n(\varphi_n x^{n+2}+\varphi_{n+2}x^n)-1\Big)\frac{x^2}{x^4+3x^2+1}$$

y aplicando (63)

$$= \frac{-x^2}{x^4+3x^2+1}$$

Y si n es impar, podemos usar (59)

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\varphi_{2k}x^{2k} = \frac{x^2(i^{n-1}(\varphi_{n-1}x^{n+1}+\varphi_{n+1}x^{n-1})-1)}{x^4+3x^2+1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\sum_{k=0}^{\infty}(-1)^k\varphi_{2k}x^{2k} = \lim_{n\to\infty}\Big(\frac{x^2(i^{n-1}(\varphi_{n-1}x^{n+1}+\varphi_{n+1}x^{n-1})-1)}{x^4+3x^2+1}\Big)$$
$$= \lim_{n\to\infty}\Big(i^{n-1}(\varphi_{n-1}x^{n+1}+\varphi_{n+1}x^{n-1})-1\Big)\frac{x^2}{x^4+3x^2+1}$$

y aplicando (62)

$$= \frac{-x^2}{x^4+3x^2+1}$$

■

Corolario 11. *Sea $k\in\mathbb{N}$, $x\in\mathbb{R}$, $-1<x<1$*

$$\sum_{k=0}^{\infty}(-1)^k\varphi_{2k+1}x^{2k+1} = \frac{x(x^2+1)}{x^4+3x^2+1} \tag{67}$$

Demostración. Sea $n\in\mathbb{N}$, $x\in\mathbb{R}$, $-1<x<1$
Si n es impar, podemos usar (60)

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\varphi_{2k+1}x^{2k+1} = \frac{x(i^{n-1}(\varphi_n x^{n+3}+\varphi_{n+2}x^{n+1})+x^2+1)}{x^4+3x^2+1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\sum_{k=0}^{\infty}(-1)^k\varphi_{2k+1}x^{2k+1} = \lim_{n\to\infty}\Big(\frac{x(i^{n-1}(\varphi_n x^{n+3}+\varphi_{n+2}x^{n+1})+x^2+1)}{x^4+3x^2+1}\Big)$$
$$= \lim_{n\to\infty}\Big(i^{n-1}(\varphi_n x^{n+3}+\varphi_{n+2}x^{n+1})+x^2+1\Big)\frac{x}{x^4+3x^2+1}$$

y aplicando (65)

$$= \frac{x(x^2+1)}{x^4+3x^2+1}$$

Y si n es par, podemos usar (61)

$$\sum_{k=0}^{\frac{n-2}{2}}(-1)^k\varphi_{2k+1}x^{2k+1} = \frac{x(i^{n-2}(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^n)+x^2+1)}{x^4+3x^2+1}$$

haciendo el límite cuando n tiende a ∞ tenemos que

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k\varphi_{2k+1}x^{2k+1} &= \lim_{n\to\infty}\Big(\frac{x(i^{n-2}(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^n)+x^2+1)}{x^4+3x^2+1}\Big)\\ &= \lim_{n\to\infty}\Big(i^{n-2}(\varphi_{n-1}x^{n+2}+\varphi_{n+1}x^n)+x^2+1\Big)\frac{x}{x^4+3x^2+1}\\ &\text{y aplicando (64)}\\ &= \frac{x(x^2+1)}{x^4+3x^2+1}\end{aligned}$$

■

2.3 Suma de los φ_{4k}

Ahora vamos a repetir nuestra construcción para la suma de los φ_{4k}, obtendremos cuatro series (68), (70), (72) y (74).

Teorema 31. *Sea $k, n, m \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\textit{Si } n = 4m,\ \sum_{k=0}^{\frac{n}{4}}\varphi_{4k}x^{4k} = \frac{x^4(\varphi_n x^{n+4}-\varphi_{n+4}x^n+3)}{x^8-7x^4+1} \tag{68}$$

$$\sum_{k=0}^{\infty}\varphi_{4k}x^{4k} = \frac{3x^4}{x^8-7x^4+1} \tag{69}$$

Demostración. Sea $n, m \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$,
$i = \sqrt{-1} \in \mathbb{C}$
Ya que $n = 4m$, podemos usar (46)

$$\sum_{k=0}^{\frac{n}{2}}\varphi_{2k}x^{2k} = \frac{x^2(\varphi_n x^{n+2}-\varphi_{n+2}x^n+1)}{x^4-3x^2+1}$$

y sumar con (58)

$$\sum_{k=0}^{\frac{n}{2}}(-1)^k\varphi_{2k}x^{2k} = \frac{x^2(i^n(\varphi_n x^{n+2}+\varphi_{n+2}x^n)-1)}{x^4+3x^2+1}$$

obtenemos

$$\begin{aligned}\sum_{k=0}^{\frac{n}{2}}\varphi_{2k}x^{2k}+\sum_{k=0}^{\frac{n}{2}}(-1)^k\varphi_{2k}x^{2k} &= \frac{x^2(\varphi_n x^{n+2}-\varphi_{n+2}x^n+1)}{x^4-3x^2+1}\\ &+\frac{x^2(i^n(\varphi_n x^{n+2}+\varphi_{n+2}x^n)-1)}{x^4+3x^2+1}\end{aligned}$$

ya que $n = 4m$ del lado izquierdo tenemos

$$\sum_{k=0}^{\frac{n}{2}}\varphi_{2k}x^{2k}+\sum_{k=0}^{\frac{n}{2}}(-1)^k\varphi_{2k}x^{2k} = \sum_{k=0}^{\frac{n}{2}}(1+(-1)^k)\varphi_{2k}x^{2k} = 2\sum_{k=0}^{\frac{n}{4}}\varphi_{4k}x^{4k}$$

y además $i^n = i^{4m} = i^{4^m} = 1^m = 1$ entonces

$$\begin{aligned}2\sum_{k=0}^{\frac{n}{4}}\varphi_{4k}x^{4k} &= \frac{x^2(\varphi_n x^{n+2}-\varphi_{n+2}x^n+1)(x^4+3x^2+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}\\ &+\frac{x^2(\varphi_n x^{n+2}+\varphi_{n+2}x^n-1)(x^4-3x^2+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}\end{aligned}$$

$$
\begin{aligned}
2\sum_{k=0}^{\frac{n}{4}}\varphi_{4k}x^{4k} = (&\varphi_n x^{n+6} - \varphi_{n+2}x^{n+4} + x^4 \\
&+3\varphi_n x^{n+4} - 3\varphi_{n+2}x^{n+2} + 3x^2 \\
&+\varphi_n x^{n+2} - \varphi_{n+2}x^n + 1 \\
&+\varphi_n x^{n+6} + \varphi_{n+2}x^{n+4} - x^4 \\
&-3\varphi_n x^{n+4} - 3\varphi_{n+2}x^{n+2} + 3x^2 \\
&+\varphi_n x^{n+2} + \varphi_{n+2}x^n - 1)\frac{x^2}{(x^4-3x^2+1)(x^4+3x^2+1)}
\end{aligned}
$$

$$
\begin{aligned}
2\sum_{k=0}^{\frac{n}{4}}\varphi_{4k}x^{4k} &= \frac{x^2(2\varphi_n x^{n+6} - 6\varphi_{n+2}x^{n+2} + 2\varphi_n x^{n+2} + 6x^2)}{(x^4-3x^2+1)(x^4+3x^2+1)} \\
\sum_{k=0}^{\frac{n}{4}}\varphi_{4k}x^{4k} &= \frac{x^2(\varphi_n x^{n+6} - 3\varphi_{n+2}x^{n+2} + \varphi_n x^{n+2} + 3x^2)}{(x^4-3x^2+1)(x^4+3x^2+1)} \\
&= \frac{x^2(\varphi_n x^{n+6} - \varphi_{n+4}x^{n+2} + 3x^2)}{(x^4-3x^2+1)(x^4+3x^2+1)} \\
&= \frac{x^4(\varphi_n x^{n+4} - \varphi_{n+4}x^n + 3)}{x^8-7x^4+1}
\end{aligned}
$$

■

Teorema 32. *Sea $k,n,m \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$
\text{Si } n = 4m+2,\ \sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k} = \frac{\varphi_n x^{n+6} - \varphi_{n+4}x^{n+2} + x^4 + 1}{x^8-7x^4+1} \tag{70}
$$

$$
\sum_{k=0}^{\infty}\varphi_{4k+2}x^{4k} = \frac{x^4+1}{x^8-7x^4+1} \tag{71}
$$

Demostración. Sea $n,m \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$,
$i = \sqrt{-1} \in \mathbb{C}$
Ya que $n = 4m+2$, podemos usar (46) y restar (58)

$$
\begin{aligned}
\sum_{k=0}^{\frac{n}{2}}\varphi_{2k}x^{2k} - \sum_{k=0}^{\frac{n}{2}}(-1)^k\varphi_{2k}x^{2k} = &\frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2}x^n + 1)}{x^4-3x^2+1} \\
&-\frac{x^2(i^n(\varphi_n x^{n+2} + \varphi_{n+2}x^n) - 1)}{x^4+3x^2+1}
\end{aligned}
$$

ya que $n = 4m+2$ del lado izquierdo tenemos

$$
\sum_{k=0}^{\frac{n}{2}}\varphi_{2k}x^{2k} - \sum_{k=0}^{\frac{n}{2}}(-1)^k\varphi_{2k}x^{2k} = \sum_{k=0}^{\frac{n}{2}}(1+(-1)^{k+1})\varphi_{2k}x^{2k} = 2\sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k+2}
$$

y además $i^n = i^{4m+2} = i^{4^m}i^2 = (1^m)(-1) = -1$ entonces

$$
\begin{aligned}
2\sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k+2} = &\frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2}x^n + 1)(x^4+3x^2+1)}{(x^4-3x^2+1)(x^4+3x^2+1)} \\
&-\frac{x^2(-\varphi_n x^{n+2} - \varphi_{n+2}x^n - 1)(x^4-3x^2+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}
\end{aligned}
$$

$$2\sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k+2}=\frac{x^2(\varphi_n x^{n+2}-\varphi_{n+2}x^n+1)(x^4+3x^2+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}$$
$$+\frac{x^2(\varphi_n x^{n+2}+\varphi_{n+2}x^n+1)(x^4-3x^2+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}$$

$$\begin{aligned}2\sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k+2}=&(\varphi_n x^{n+6}-\varphi_{n+2}x^{n+4}+x^4\\&+3\varphi_n x^{n+4}-3\varphi_{n+2}x^{n+2}+3x^2\\&+\varphi_n x^{n+2}-\varphi_{n+2}x^n+1\\&+\varphi_n x^{n+6}+\varphi_{n+2}x^{n+4}+x^4\\&-3\varphi_n x^{n+4}-3\varphi_{n+2}x^{n+2}-3x^2\\&+\varphi_n x^{n+2}+\varphi_{n+2}x^n+1)\frac{x^2}{(x^4-3x^2+1)(x^4+3x^2+1)}\end{aligned}$$

$$2\sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k+2}=\frac{x^2(2\varphi_n x^{n+6}-6\varphi_{n+2}x^{n+2}+2\varphi_n x^{n+2}+2x^4+2)}{(x^4-3x^2+1)(x^4+3x^2+1)}$$
$$\sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k+2}=\frac{x^2(\varphi_n x^{n+6}-3\varphi_{n+2}x^{n+2}+\varphi_n x^{n+2}+x^4+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}$$
$$\sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k+2}=\frac{x^2(\varphi_n x^{n+6}-\varphi_{n+4}x^{n+2}+x^4+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}$$
$$\sum_{k=0}^{\frac{n-2}{4}}\varphi_{4k+2}x^{4k+2}=\frac{x^2(\varphi_n x^{n+6}-\varphi_{n+4}x^{n+2}+x^4+1)}{x^8-7x^4+1}$$

■

Teorema 33. *Sea* $k,n,m\in\mathbb{N}$*,* $x\in\mathbb{R}$*,* $-1<x<1$*,* $x\neq\{\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$

$$\text{Si } n=4m+1,\ \sum_{k=0}^{\frac{n-1}{4}}\varphi_{4k}x^{4k}=\frac{x^4(\varphi_{n-1}x^{n+3}-\varphi_{n+3}x^{n-1}+3)}{x^8-7x^4+1}\tag{72}$$

$$\sum_{k=0}^{\infty}\varphi_{4k}x^{4k}=\frac{3x^4}{x^8-7x^4+1}\tag{73}$$

Demostración. Sea $n,m\in\mathbb{N}$, $x\in\mathbb{R}$, $-1<x<1$, $x\neq\{\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$,
$i=\sqrt{-1}\in\mathbb{C}$
Ya que $n=4m+1$, podemos usar (47)

$$\sum_{k=0}^{\frac{n-1}{2}}\varphi_{2k}x^{2k}=\frac{x^2(\varphi_{n-1}x^{n+1}-\varphi_{n+1}x^{n-1}+1)}{x^4-3x^2+1}$$

y sumar con (59)

$$\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\varphi_{2k}x^{2k}=\frac{x^2(i^{n-1}(\varphi_{n-1}x^{n+1}+\varphi_{n+1}x^{n-1})-1)}{x^4+3x^2+1}$$

obtenemos

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} + \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \varphi_{2k}x^{2k} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1)}{x^4 - 3x^2 + 1} + \frac{x^2(i^{n-1}(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1}) - 1)}{x^4 + 3x^2 + 1}$$

ya que $n = 4m + 1$ del lado izquierdo tenemos

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k}x^{2k} + \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \varphi_{2k}x^{2k} = \sum_{k=0}^{\frac{n-1}{2}} (1 + (-1)^k)\varphi_{2k}x^{2k} = 2\sum_{k=0}^{\frac{n-1}{4}} \varphi_{4k}x^{4k}$$

y además $i^{n-1} = i^{4m} = i^{4^m} = 1^m = 1$ entonces

$$2\sum_{k=0}^{\frac{n-1}{4}} \varphi_{4k}x^{4k} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1)(x^4 + 3x^2 + 1)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)} + \frac{x^2(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1} - 1)(x^4 - 3x^2 + 1)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}$$

$$\begin{aligned}2\sum_{k=0}^{\frac{n-1}{4}} \varphi_{4k}x^{4k} = (&\varphi_{n-1}x^{n+5} - \varphi_{n+1}x^{n+3} + x^4 \\ &+3\varphi_{n-1}x^{n+3} - 3\varphi_{n+1}x^{n+1} + 3x^2 \\ &+\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1 \\ &+\varphi_{n-1}x^{n+5} + \varphi_{n+1}x^{n+3} - x^4 \\ &-3\varphi_{n-1}x^{n+3} - 3\varphi_{n+1}x^{n+1} + 3x^2 \\ &+\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1} - 1)\frac{x^2}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}\end{aligned}$$

$$2\sum_{k=0}^{\frac{n-1}{4}} \varphi_{4k}x^{4k} = \frac{x^2(2\varphi_{n-1}x^{n+5} - 6\varphi_{n+1}x^{n+1} + 2\varphi_{n-1}x^{n+1} + 6x^2)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}$$

$$\sum_{k=0}^{\frac{n-1}{4}} \varphi_{4k}x^{4k} = \frac{x^2(\varphi_{n-1}x^{n+5} - 3\varphi_{n+1}x^{n+1} + \varphi_{n-1}x^{n+1} + 3x^2)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}$$

$$\sum_{k=0}^{\frac{n-1}{4}} \varphi_{4k}x^{4k} = \frac{x^2(\varphi_{n-1}x^{n+5} - \varphi_{n+3}x^{n+1} + 3x^2)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}$$

$$\sum_{k=0}^{\frac{n-1}{4}} \varphi_{4k}x^{4k} = \frac{x^4(\varphi_{n-1}x^{n+3} - \varphi_{n+3}x^{n-1} + 3)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}$$

$$\sum_{k=0}^{\frac{n-1}{4}} \varphi_{4k}x^{4k} = \frac{x^4(\varphi_{n-1}x^{n+3} - \varphi_{n+3}x^{n-1} + 3)}{x^8 - 7x^4 + 1}$$

■

Teorema 34. *Sea* $k, n, m \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\text{Si } n = 4m + 3, \ \sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2}x^{4k} = \frac{\varphi_{n-1}x^{n+5} - \varphi_{n+3}x^{n+1} + x^4 + 1}{x^8 - 7x^4 + 1} \tag{74}$$

$$\sum_{k=0}^{\infty} \varphi_{4k+2} x^{4k} = \frac{x^4+1}{x^8-7x^4+1} \tag{75}$$

Demostración. Ya que $n = 4m+3$, podemos usar (47) y restar con (59)

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k} x^{2k} - \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \varphi_{2k} x^{2k} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1}+1)}{x^4-3x^2+1}$$
$$- \frac{x^2(i^{n-1}(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1}) - 1)}{x^4+3x^2+1}$$

ya que $n = 4m+3$ del lado izquierdo tenemos

$$\sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k} x^{2k} - \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \varphi_{2k} x^{2k} = \sum_{k=0}^{\frac{n-1}{2}} (1+(-1)^{k+1}) \varphi_{2k} x^{2k} = 2\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2} x^{4k+2}$$

y además $i^{n-1} = i^{4m+2} = i^{4m} i^2 = (1^m)(-1) = -1$ entonces

$$2\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2} x^{4k+2} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1}+1)}{x^4-3x^2+1}$$
$$- \frac{x^2(-\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} - 1)}{x^4+3x^2+1}$$

$$2\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2} x^{4k+2} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1}+1)}{x^4-3x^2+1}$$
$$+ \frac{x^2(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1} + 1)}{x^4+3x^2+1}$$

$$2\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2} x^{4k+2} = \frac{x^2(\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1}+1)(x^4+3x^2+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}$$
$$+ \frac{x^2(\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1} + 1)(x^4-3x^2+1)}{(x^4-3x^2+1)(x^4+3x^2+1)}$$

$$2\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2} x^{4k+2} = (\varphi_{n-1}x^{n+5} - \varphi_{n+1}x^{n+3} + x^4$$
$$+3\varphi_{n-1}x^{n+3} - 3\varphi_{n+1}x^{n+1} + 3x^2$$
$$+\varphi_{n-1}x^{n+1} - \varphi_{n+1}x^{n-1} + 1$$
$$+\varphi_{n-1}x^{n+5} + \varphi_{n+1}x^{n+3} + x^4$$
$$-3\varphi_{n-1}x^{n+3} - 3\varphi_{n+1}x^{n+1} - 3x^2$$
$$+\varphi_{n-1}x^{n+1} + \varphi_{n+1}x^{n-1} + 1)\frac{x^2}{(x^4-3x^2+1)(x^4+3x^2+1)}$$

$$2\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2}x^{4k+2} = \frac{x^2(2\varphi_{n-1}x^{n+5} - 6\varphi_{n+1}x^{n+1} + 2\varphi_{n-1}x^{n+1} + 2x^4 + 2)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}$$
$$\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2}x^{4k+2} = \frac{x^2(\varphi_{n-1}x^{n+5} - 3\varphi_{n+1}x^{n+1} + \varphi_{n-1}x^{n+1} + x^4 + 1)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}$$
$$\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2}x^{4k+2} = \frac{x^2(\varphi_{n-1}x^{n+5} - \varphi_{n+3}x^{n+1} + x^4 + 1)}{(x^4 - 3x^2 + 1)(x^4 + 3x^2 + 1)}$$
$$\sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2}x^{4k+2} = \frac{x^2(\varphi_{n-1}x^{n+5} - \varphi_{n+3}x^{n+1} + x^4 + 1)}{x^8 - 7x^4 + 1}$$

■

2.4 Suma de los φ_{8k}

Finalmente construiremos la suma de los φ_{8k}, (76) y (77).

Teorema 35. *Sea $k, n, m \in \mathbb{N}$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\text{Si } n = 8m,\ \sum_{k=0}^{\frac{n}{8}} \varphi_{8k}x^{8k} = \frac{x^8(\varphi_n x^{n+8} + (\varphi_n - 7\varphi_{n+4})x^n + 21)}{x^{16} - 47x^8 + 1} \tag{76}$$

$$\sum_{k=0}^{\infty} \varphi_{8k}x^{8k} = \frac{21x^8}{x^{16} - 47x^8 + 1} \tag{77}$$

Demostración. Sea $n = 8m$, entonces

$$\sum_{k=0}^{\frac{n}{4}} \varphi_{4k}x^{4k} = \frac{x^4(\varphi_n x^{n+4} - \varphi_{n+4}x^n + 3)}{x^8 - 7x^4 + 1}$$

Sustituyendo $x = (-1)^{\frac{1}{4}}x$

$$\sum_{k=0}^{\frac{n}{4}} (-1)^k \varphi_{4k}x^{4k} = \frac{x^4(\varphi_n x^{n+4} + \varphi_{n+4}x^n - 3)}{x^8 + 7x^4 + 1}$$

Y sumando obtenemos

$$\sum_{k=0}^{\frac{n}{4}} \varphi_{4k}x^{4k} + \sum_{k=0}^{\frac{n}{4}} (-1)^k \varphi_{4k}x^{4k} = \sum_{k=0}^{\frac{n}{4}} (1 + (-1)^k)\varphi_{4k}x^{4k} = 2\sum_{k=0}^{\frac{n}{8}} \varphi_{8k}x^{8k}$$
$$2\sum_{k=0}^{\frac{n}{8}} \varphi_{8k}x^{8k} = \frac{x^4(\varphi_n x^{n+4} - \varphi_{n+4}x^n + 3)(x^8 + 7x^4 + 1)}{(x^8 - 7x^4 + 1)(x^8 + 7x^4 + 1)} + \frac{x^4(\varphi_n x^{n+4} + \varphi_{n+4}x^n - 3)(x^8 - 7x^4 + 1)}{(x^8 - 7x^4 + 1)(x^8 + 7x^4 + 1)}$$

$$\begin{aligned}
2\sum_{k=0}^{\frac{n}{8}} \varphi_{8k}x^{8k} = (&\varphi_n x^{n+12} - \varphi_{n+4}x^{n+8} + 3x^8 \\
&+ 7\varphi_n x^{n+8} - 7\varphi_{n+4}x^{n+4} + 21x^4 \\
&+ \varphi_n x^{n+4} - \varphi_{n+4}x^n + 3 \\
&+ \varphi_n x^{n+12} + \varphi_{n+4}x^{n+8} - 3x^8 \\
&- 7\varphi_n x^{n+8} - 7\varphi_{n+4}x^{n+4} + 21x^4 \\
&+ \varphi_n x^{n+4} + \varphi_{n+4}x^n - 3)\frac{x^4}{(x^8 - 7x^4 + 1)(x^8 + 7x^4 + 1)} \\
\sum_{k=0}^{\frac{n}{8}} \varphi_{8k}x^{8k} &= \frac{x^4(\varphi_n x^{n+12} - 7\varphi_{n+4}x^{n+4} + 21x^4 + \varphi_n x^{n+4})}{(x^8 - 7x^4 + 1)(x^8 + 7x^4 + 1)} \\
\sum_{k=0}^{\frac{n}{8}} \varphi_{8k}x^{8k} &= \frac{x^8(\varphi_n x^{n+8} + (\varphi_n - 7\varphi_{n+4})x^n + 21)}{(x^8 - 7x^4 + 1)(x^8 + 7x^4 + 1)} \\
\sum_{k=0}^{\frac{n}{8}} \varphi_{8k}x^{8k} &= \frac{x^8(\varphi_n x^{n+8} + (\varphi_n - 7\varphi_{n+4})x^n + 21)}{x^{16} - 47x^8 + 1}
\end{aligned}$$

■

2.5 Sucesión β

Sea $k \in \mathbb{N}$, definimos la siguiente sucesión:

$$\beta_{2^k} = \psi^{2^k} + \tau^{2^k} = \left(\frac{1+\sqrt{5}}{2}\right)^{2^k} + \left(\frac{1-\sqrt{5}}{2}\right)^{2^k}$$

$$\begin{aligned}
k = 0, &\quad \beta_1 = 1 \\
k = 1, &\quad \beta_2 = (\beta_1)^2 + 2 = 1^2 + 2 = 3 \\
k = 2, &\quad \beta_4 = (\beta_2)^2 - 2 = 3^2 - 2 = 7 \\
k = 3, &\quad \beta_8 = (\beta_4)^2 - 2 = 7^2 - 2 = 47
\end{aligned}$$

Teorema 36. *Sea $k, m, r \in \mathbb{N}$ y $r = 2^m, m \geqslant 2$ entonces*

$$\prod_{k=0}^{m-1} \beta_{2^k} = \varphi_r \tag{78}$$

2.6 Las r-particiones

Teorema 37. *Sea $k, n, m, r, s \in \mathbb{N}$, $m \geqslant 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $\alpha \in \mathbb{Z}[x]$ Y sea $(n - s) \equiv 0 \pmod{r}$*

$$Si\ (s < \frac{r}{2}) \Rightarrow \sum_{k=0}^{\frac{n-s}{r}} \varphi_{rk}x^{rk} = \frac{x^n\alpha + \varphi_r x^r}{x^{2r} - \beta_r x^r + 1} \tag{79}$$

Teorema 38. *Sea $k, n, m, r, s \in \mathbb{N}$, $m > 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $\alpha \in \mathbb{Z}[x]$ Y sea $(n - s) \equiv 0 \pmod{r}$*

$$Si\ (s \geqslant \frac{r}{2}) \Rightarrow \sum_{k=0}^{\frac{n-s}{r}} \varphi_{rk+\frac{r}{2}}x^{rk} = \frac{x^n\alpha + \varphi_{\frac{r}{2}}(x^r + 1)}{x^{2r} - \beta_r x^r + 1} \tag{80}$$

Corolario 12. *Sea* $k, n, m, r, s \in \mathbb{N}$, $m \geqslant 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$, $\alpha \in \mathbb{Z}[x]$
Y sea $(n - s) \equiv 0 \pmod{r}$

$$Si\ (s < \frac{r}{2}) \Rightarrow \sum_{k=0}^{\frac{n-s}{r}} (-1)^k \varphi_{rk} x^{rk} = \frac{x^n \alpha - \varphi_r x^r}{x^{2r} + \beta_r x^r + 1} \tag{81}$$

Corolario 13. *Sea* $k, n, m, r, s \in \mathbb{N}$, $m > 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$, $\alpha \in \mathbb{Z}[x]$
Y sea $(n - s) \equiv 0 \pmod{r}$

$$Si\ (s \geqslant \frac{r}{2}) \Rightarrow \sum_{k=0}^{\frac{n-s}{r}} (-1)^k \varphi_{rk+\frac{r}{2}} x^{rk} = \frac{x^n \alpha + \varphi_{\frac{r}{2}}(1 - x^r)}{x^{2r} + \beta_r x^r + 1} \tag{82}$$

Demostración. Tenemos que a partir de la siguiente serie finita

$$\sum_{k=0}^{n} \varphi_k x^k = \frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2 + x - 1}$$

si hacemos $x := -x$ obtenemos la serie alternada

$$\sum_{k=0}^{n} (-1)^k \varphi_k x^k = \frac{(-1)^n \varphi_n x^{n+2} + (-1)^{n+1} \varphi_{n+1} x^{n+1} + x}{x^2 - x - 1}$$

y podemos sumarlas para el caso par y para el caso impar y obtener

$$\text{Si } n \text{ es par, } \sum_{k=0}^{\frac{n}{2}} \varphi_{2k} x^{2k} = \frac{x^2(\varphi_n x^{n+2} - \varphi_{n+2} x^n + 1)}{x^4 - 3x^2 + 1}$$

$$\text{Y si } n \text{ es impar, } \sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k} x^{2k} = \frac{x^2(\varphi_{n-1} x^{n+1} - \varphi_{n+1} x^{n-1} + 1)}{x^4 - 3x^2 + 1}$$

si nos fijamos en el polinomio del denominador

$$x^4 - 3x^2 + 1 = 0$$

esta igualdad se cumple para x igual a $\psi = \left|\frac{1+\sqrt{5}}{2}\right|$, $\tau = \left|\frac{1-\sqrt{5}}{2}\right|$.
Además para $r \geqslant 2$ esta forma del polinomio

$$x^{2r} - bx^r + 1 = 0$$

ya no cambia en los siguientes pasos recursivos y haciendo $x := (-1)^{1/r} x$ podemos generar el denominador de la serie alternada

$$x^{2r} + bx^r + 1 = 0$$

si las multiplicamos obtenemos

$$\begin{aligned}
(x^{2r} - bx^r + 1)(x^{2r} + bx^r + 1) &= (x^{4r} + bx^{3r} + x^{2r}) - (bx^{3r} + b^2 x^{2r} + bx^r) + (x^{2r} + bx^r + 1) \\
&= x^{4r} + 2x^{2r} - b^2 x^{2r} + 1 \\
&= x^{4r} + (2 - b^2) x^{2r} + 1 \\
&= x^{4r} - (b^2 - 2) x^{2r} + 1,
\end{aligned}$$

es decir, que $b_{n+1} = b_n^2 - 2$ y en donde $b_2 = 3$, con esto podemos ver que b es una función que depende de r

$$b(2r) = b(r)^2 - 2$$

Ahora bien, si hacemos $r = 1$ en el polinomio tenemos que

$$x^2 - bx + 1 = 0, \quad x = \frac{b \pm \sqrt{b^2 - 4}}{2}$$

y podemos elevar al cuadrado x y obtener

$$x^2 = \Big(\frac{b \pm \sqrt{b^2 - 4}}{2}\Big)^2 = \frac{2b^2 - 4 \pm 2b\sqrt{b^2 - 4}}{4} = \frac{b^2 - 2 \pm b\sqrt{b^2 - 4}}{2},$$

que satisface para el caso $r = 2$ del polinomio

$$x^4 - bx^2 + 1 = 0$$

Entonces ya que $b_2 = 3$ tenemos que

$$\begin{aligned} b^2 - 2 &= 3 \\ b^2 &= 5 \\ b &= \sqrt{5} \end{aligned}$$

Entonces x^2 y x son

$$x^2 = \frac{3 \pm \sqrt{5}}{2}, \quad x = \Big|\frac{1 \pm \sqrt{5}}{2}\Big|$$

Sea $r \in \mathbb{N}$, definimos β como

$$\beta(r) = \psi^r + \tau^r = \Big(\frac{1 + \sqrt{5}}{2}\Big)^r + \Big(\frac{1 - \sqrt{5}}{2}\Big)^r$$

la cual cumple con la propiedad de $\beta(2r) = \beta(r)^2 - 2$ y además cumple con $b(2) = 3$ y sus primeras evaluaciones:

$$\beta(0) = 2, \quad \beta(1) = 1, \quad \beta(2) = 3, \quad \beta(4) = 7, \quad \beta(8) = 47.$$

Para la parte del numerador tenemos dos casos uno para la suma:

$$\begin{aligned} 2\sum_{k=0}^{\frac{n}{2r}} \varphi_{2rk} x^{2rk} &= \sum_{k=0}^{\frac{n}{r}} \varphi_{rk} x^{rk} + \sum_{k=0}^{\frac{n}{r}} (-1)^k \varphi_{rk} x^{rk} = \frac{x^n\alpha + \varphi_r x^r}{x^{2r} - \beta_r x^r + 1} + \frac{x^n\gamma - \varphi_r x^r}{x^{2r} + \beta_r x^r + 1} \\ &= \frac{(x^n\alpha + \varphi_r x^r)(x^{2r} + \beta_r x^r + 1) + (x^n\gamma - \varphi_r x^r)(x^{2r} - \beta_r x^r + 1)}{x^{4r} - \beta_{2r} x^{2r} + 1} \\ &= \frac{x^n\delta + (\varphi_r x^{3r} + \varphi_r\beta_r x^{2r} + \varphi_r x^r) + (-\varphi_r x^{3r} + \varphi_r\beta_r x^{2r} - \varphi_r x^r)}{x^{4r} - \beta_{2r} x^{2r} + 1} \\ \sum_{k=0}^{\frac{n}{2r}} \varphi_{2rk} x^{2rk} &= \frac{x^n\delta' + \varphi_r\beta_r x^{2r}}{x^{4r} - \beta_{2r} x^{2r} + 1} \end{aligned}$$

Aquí podemos fijarnos nuevamente en nuestro caso base y elevar al cuadrado

$$\Big(\frac{\beta(2) - \beta(1)\sqrt{5}}{2}\Big)^2 = \Big(\frac{3 - \sqrt{5}}{2}\Big)^2 = \frac{14 - 6\sqrt{5}}{4} = \frac{7 - 3\sqrt{5}}{2} = \frac{\beta(4) - \beta(2)\beta(1)\sqrt{5}}{2}$$

y en general tenemos que

$$\Big(\frac{\beta(r) - p\sqrt{5}}{2}\Big)^2 = \frac{\beta(r)^2 + 5p^2 - 2p\beta(r)\sqrt{5}}{4} = \frac{(\beta(r)^2 + 5p^2)/2 - p\beta(r)\sqrt{5}}{2}$$

Y ya que φ cumple con

$$\Big(\frac{1 + \sqrt{5}}{2}\Big)^r - \Big(\frac{1 - \sqrt{5}}{2}\Big)^r = \varphi(r)\sqrt{5}$$

podemos concluir que si $r = 2^m$, $m \geqslant 1$, entonces

$$\varphi(r)\beta(r) = \varphi(2r), \qquad \prod_{k=0}^{m-1} \beta(2^k) = \varphi(r)$$

$$\varphi(r) = \sqrt{\frac{\beta(r)^2 - 4}{5}}, \qquad \beta(r) = \sqrt{5\varphi(r)^2 + 4}$$

Y con esto tenemos que, sea $k, r, n, m \in \mathbb{N}$, $r = 2^m$, $m \geqslant 1$, $\alpha \in \mathbb{Z}[x]$,

$$\sum_{k=0}^{\frac{n}{r}} \varphi_{rk} x^{rk} = \frac{x^n \alpha + \varphi_r x^r}{x^{2r} - \beta_r x^r + 1}$$

El otro caso resulta restando las series

$$2\sum_{k=0}^{\frac{n-r}{2r}} \varphi_{2rk+r} x^{2rk+r} = \sum_{k=0}^{\frac{n}{r}} \varphi_{rk} x^{rk} - \sum_{k=0}^{\frac{n}{r}} (-1)^k \varphi_{rk} x^{rk} = \frac{x^n \alpha + \varphi_r x^r}{x^{2r} - \beta_r x^r + 1} - \frac{x^n \gamma - \varphi_r x^r}{x^{2r} + \beta_r x^r + 1}$$

$$2\sum_{k=0}^{\frac{n-r}{2r}} \varphi_{2rk+r} x^{2rk} = \frac{x^{n-r}\alpha + \varphi_r}{x^{2r} - \beta_r x^r + 1} - \frac{x^{n-r}\gamma - \varphi_r}{x^{2r} + \beta_r x^r + 1}$$

$$= \frac{(x^{n-r}\alpha + \varphi_r)(x^{2r} + \beta_r x^r + 1) + (x^{n-r}\gamma' + \varphi_r)(x^{2r} - \beta_r x^r + 1)}{x^{4r} - \beta_{2r} x^{2r} + 1}$$

$$= \frac{x^{n-r}\delta + (\varphi_r x^{2r} + \varphi_r \beta_r x^r + \varphi_r) + (\varphi_r x^{2r} - \varphi_r \beta_r x^r + \varphi_r)}{x^{4r} - \beta_{2r} x^{2r} + 1}$$

$$\sum_{k=0}^{\frac{n-r}{2r}} \varphi_{2rk+r} x^{2rk} = \frac{x^n \delta' + \varphi_r (x^{2r} + 1)}{x^{4r} - \beta_{2r} x^{2r} + 1}$$

Entonces, sea $k, r, r', n, m \in \mathbb{N}$, $r = 2^m$, $r' = \frac{r}{2}$, $\alpha \in \mathbb{Z}[x]$,

$$\sum_{k=0}^{\frac{n-r'}{r}} \varphi_{rk+r'} x^{rk} = \frac{x^n \alpha + \varphi_{r'}(x^r + 1)}{x^{2r} - \beta_r x^r + 1}$$

Únicamente falta por demostrar a partir de m se cumple, entonces sabiendo que el $-x$ en la serie original

$$\sum_{k=0}^{n} \varphi_k x^k = \frac{\varphi_n x^{n+2} + \varphi_{n+1} x^{n+1} - x}{x^2 + x - 1}$$

al sumar con la alternante resulta en

$$\text{Si } n \text{ es impar, } \sum_{k=0}^{\frac{n-1}{2}} \varphi_{2k+1} x^{2k+1} = \frac{x(\varphi_n x^{n+3} - \varphi_{n+2} x^{n+1} - x^2 + 1)}{x^4 - 3x^2 + 1}$$

$$\text{Y si } n \text{ es par, } \sum_{k=0}^{\frac{n-2}{2}} \varphi_{2k+1} x^{2k+1} = \frac{x(\varphi_{n-1} x^{n+2} - \varphi_{n+1} x^{n} - x^2 + 1)}{x^4 - 3x^2 + 1}$$

y se genera un $-x^2$, y a partir del siguiente caso en donde $r = 4$ el signo se corrige siempre a positivo:

$$\text{Si } n = 4m + 2, \sum_{k=0}^{\frac{n-2}{4}} \varphi_{4k+2} x^{4k} = \frac{\varphi_n x^{n+6} - \varphi_{n+4} x^{n+2} + x^4 + 1}{x^8 - 7x^4 + 1}$$

$$\text{Si } n = 4m + 3, \sum_{k=0}^{\frac{n-3}{4}} \varphi_{4k+2} x^{4k} = \frac{\varphi_{n-1} x^{n+5} - \varphi_{n+3} x^{n+1} + x^4 + 1}{x^8 - 7x^4 + 1}$$

y podemos decir que a partir de $m > 1$.
Por último, en cada iteración del proceso constructivo podemos sumar y restar $\frac{r}{2}$ veces fórmulas.
Entonces si hacemos correr s sobre los naturales y se cumple con

$$n \equiv s \pmod r$$

tendremos $0 \leqslant s < \frac{r}{2}$ sumas:

$$\sum_{k=0}^{\frac{n-s}{r}} \varphi_{rk} x^{rk} = \frac{x^n \alpha + \varphi_r x^r}{x^{2r} - \beta_r x^r + 1}$$

y $\frac{r}{2} \leqslant s < r$ restas:

$$\sum_{k=0}^{\frac{n-s}{r}} \varphi_{rk+r'} x^{rk} = \frac{x^n \alpha + \varphi_{r'}(x^r + 1)}{x^{2r} - \beta_r x^r + 1}$$

■

2.7 Límites A,B,C,D

Teorema 39 (Límite A). *Sea $k, m, r \in \mathbb{N}$, $m \geqslant 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} = \frac{\varphi_r x^r}{x^{2r} - \beta_r x^r + 1} \tag{83}$$

Teorema 40 (Límite B). *Sea $k, m, r \in \mathbb{N}$, $m > 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$, $x \neq \{\frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\sum_{k=0}^{\infty} \varphi_{rk+\frac{r}{2}} x^{rk} = \frac{\varphi_{\frac{r}{2}}(x^r + 1)}{x^{2r} - \beta_r x^r + 1} \tag{84}$$

Corolario 14 (Límite C). *Sea $k, m, r \in \mathbb{N}$, $m \geqslant 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$*

$$\sum_{k=0}^{\infty} (-1)^k \varphi_{rk} x^{rk} = \frac{-\varphi_r x^r}{x^{2r} + \beta_r x^r + 1} \tag{85}$$

Corolario 15 (Límite D). *Sea $k, m, r \in \mathbb{N}$, $m > 1$, $r = 2^m$, $x \in \mathbb{R}$, $-1 < x < 1$*

$$\sum_{k=0}^{\infty} (-1)^k \varphi_{rk+\frac{r}{2}} x^{rk} = \frac{\varphi_{\frac{r}{2}}(1 - x^r)}{x^{2r} + \beta_r x^r + 1} \tag{86}$$

3 Ecuaciones diferenciales

3.1 Separación lineal, $r \geqslant 4$

Sabemos por la naturaleza de la fórmula de Binet (28) que la sucesión de fibonacci se puede extender a todos los números reales, tal que para todo $r \in \mathbb{R}$ la siguiente función esta bien definida:

$$\varphi_r = \frac{\psi^r - \tau^r}{\sqrt{5}},$$

sin embargo nosotros queremos preguntarnos acerca de la suma tal y como actúa el Límite A. De tal manera que nuestra suma infinita (83) se convierte en una integral (89), y la función de fibonacci se extiende a dos variables sobre los reales (92). La parte lineal en r que se acumulaba en el φ_r se traduce ahora en una función derivable Ψ. Sea $r, x \in \mathbb{R}$, $r \geqslant 4$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $\psi = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$.
Definimos β de la siguiente manera

$$\beta(r) = \psi^r + \tau^r = \Big(\frac{1+\sqrt{5}}{2}\Big)^r + \Big(\frac{1-\sqrt{5}}{2}\Big)^r \tag{87}$$

Y definimos Θ de la siguiente manera

$$\Theta(x, r) = \frac{x^r}{x^{2r} - \beta(r)x^r + 1} \tag{88}$$

Y sea Ψ una función derivable en r y $\alpha, b \in \mathbb{R}$, entonces podemos definir Ω de la siguiente manera

$$\Omega(x, r) = \int \Phi(x, r)dr = \alpha\Psi(r)\Theta(x, r) + b \tag{89}$$

Derivando β con respecto a r obtenemos

$$(\beta(r))' = \psi^r \log\psi + \tau^r \log\tau = \log\psi^{\psi^r} + \log\tau^{\tau^r} \tag{90}$$

Derivando Θ con respecto a r obtenemos

$$\begin{aligned}
(\Theta(x,r))' &= \frac{(x^r)'(x^{2r} - \beta(r)x^r + 1) - (x^{2r} - \beta(r)x^r + 1)'(x^r)}{(x^{2r} - \beta(r)x^r + 1)^2} \\
&= \frac{(x^r \log x)(x^{2r} - \beta(r)x^r + 1) - (2x^{2r}\log x - (\beta(r))'x^r - \beta(r)x^r \log x)(x^r)}{(x^{2r} - \beta(r)x^r + 1)^2} \\
&= \frac{(x^{3r}\log x - \beta(r)x^{2r}\log x + x^r \log x) - (2x^{3r}\log x - (\beta(r))'x^{2r} - \beta(r)x^{2r}\log x)}{(x^{2r} - \beta(r)x^r + 1)^2}
\end{aligned}$$

$$(\Theta(x,r))' = \frac{(\beta(r))'x^{2r} + x^r \log x - x^{3r}\log x}{(x^{2r} - \beta(r)x^r + 1)^2} \tag{91}$$

Por tanto

$$\begin{aligned}
\frac{(\Omega(x,r))'}{\alpha} = \frac{\Phi(x,r)}{\alpha} &= \Psi(r)\frac{(\beta(r))'x^{2r} + x^r \log x - x^{3r}\log x}{(x^{2r} - \beta(r)x^r + 1)^2} + (\Psi(r))'\frac{x^r}{x^{2r} - \beta(r)x^r + 1} \\
&= \Psi(r)\frac{((\beta(r))' + \frac{\log x}{x^r} - x^r \log x)x^{2r}}{(x^{2r} - \beta(r)x^r + 1)^2} + (\Psi(r))'\frac{x^r}{x^{2r} - \beta(r)x^r + 1} \\
&= \Psi(r)\Theta^2(x,r)\big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\big) + (\Psi(r))'\Theta(x,r)
\end{aligned}$$

$$\Phi(x,r) = \alpha\,\Theta^2(x,r)\Big(\Psi(r)\big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\big) + \frac{(\Psi(r))'}{\Theta(x,r)}\Big) \tag{92}$$

Y tenemos que

$$(\beta(r))' + \frac{\log x}{x^r} - x^r \log x = \log\psi^{\psi^r} + \log\tau^{\tau^r} + \log x^{x^{-r}} - \log x^{x^r} \tag{93}$$

Y además

$$\alpha\,\Psi(r) = \frac{1}{\Theta(x,r)}\Big(\int \Phi(x,r)dr - b\Big)$$

En particular podemos fijarnos en $x = 1$, y tenemos que Θ cumple con

$$\Theta(1,r) = \frac{1}{2-\beta(r)}, \quad \Theta(1,2r) = \frac{1}{2-\beta(2r)}$$

Entonces queremos que Φ cumpla con

$$\Phi(1,r) = \varphi_r = \frac{\psi^r - \tau^r}{\sqrt{5}}$$

Definimos $\sigma = \psi^2 = \frac{3+\sqrt{5}}{2}$, $\upsilon = \tau^2 = \frac{3-\sqrt{5}}{2}$, tenemos que

$$\Phi(1,2r) = \varphi_{2r} = \frac{\psi^{2r} - \tau^{2r}}{\sqrt{5}} = \frac{\sigma^r - \upsilon^r}{\sqrt{5}}$$

Por tanto

$$\begin{aligned}
\alpha\Psi(2r) &= (2-\beta(2r))\Big(\int \Big(\frac{\sigma^r - \upsilon^r}{\sqrt{5}}\Big)d(2r) - b\Big)\\
&= \frac{2(2-\beta(2r))}{\sqrt{5}}\Big(\int \big(\sigma^r - \upsilon^r\big)dr - b\Big)\\
&= \frac{2(2-\beta(2r))}{\sqrt{5}}\Big(\frac{\sigma^r}{\log\sigma} - \frac{\upsilon^r}{\log\upsilon} - b\Big)
\end{aligned}$$

Y entonces

$$\Psi(r) = (2-\beta(r))\Big(\frac{\psi^r}{\log\psi} - \frac{\tau^r}{\log\tau} - b\Big), \qquad \alpha = \frac{1}{\sqrt{5}} \tag{94}$$

Y en particular cumple para $r = 4$ y tenemos que

$$\begin{aligned}
3 = \varphi(4) = \Psi(4) &= (2-\beta(4))\Big(\frac{\psi^4}{\log\psi} - \frac{\tau^4}{\log\tau} - b\Big)\\
3 &= (2-7)\Big(\frac{\psi^4}{\log\psi} - \frac{\tau^4}{\log\tau} - b\Big)\\
-\frac{3}{5} &= \frac{\psi^4}{\log\psi} - \frac{\tau^4}{\log\tau} - b
\end{aligned}$$

Y tenemos que podemos tomar b como constante

$$b = \frac{\psi^4}{\log\psi} - \frac{\tau^4}{\log\tau} + \frac{3}{5} \tag{95}$$

Entonces tenemos que Ω es igual

$$\Omega(x,r) = \int \Phi(x,r)dr = \frac{(2-\beta(r))}{\sqrt{5}}\Big(\frac{\psi^r}{\log\psi} - \frac{\tau^r}{\log\tau} - b\Big)\frac{x^r}{x^{2r} - \beta(r)x^r + 1} \tag{96}$$

Y derivando Ψ tenemos

$$\begin{aligned}
(\Psi(r))' &= -(\beta(r))'\Big(\frac{\psi^r}{\log\psi} - \frac{\tau^r}{\log\tau} - b\Big) + (2-\beta(r))(\psi^r - \tau^r)\\
&= (2-\beta(r))\Big((\psi^r - \tau^r) - \frac{(\beta(r))'}{(2-\beta(r))}\Big(\frac{\psi^r}{\log\psi} - \frac{\tau^r}{\log\tau} - b\Big)\Big)\\
&= (2-\beta(r))\Big((\psi^r - \tau^r) - \frac{(\beta(r))'}{(2-\beta(r))^2}\Psi(r)\Big)
\end{aligned}$$

De tal manera que φ_r

$$\varphi_r = \frac{(\psi^r - \tau^r)}{\sqrt{5}} = \frac{1}{\sqrt{5}}\Big(\frac{(\Psi(r))'}{(2-\beta(r))} + \frac{(\beta(r))'\Psi(r)}{(2-\beta(r))^2}\Big) \tag{97}$$

3.2 Caso $1 < r < 4$

Adicional tenemos que para $1 < r < 4$ se cumple con

$$\int \Phi(x,r)dr = \gamma\Theta(x,r)$$

Entonces haciendo $x = 1$ tenemos que

$$(\Theta(1,r))' = \frac{(\beta(r))'}{(2-\beta(r))^2}$$

Evaluando en $r = 2$ tenemos

$$\begin{aligned}
(\Theta(1,2))' &= \frac{\sqrt{5}}{2}\log\big(\frac{3+\sqrt{5}}{2}\big)\\
1 = \varphi(2) &= \gamma_1\frac{\sqrt{5}}{2}\log\big(\frac{3+\sqrt{5}}{2}\big)\\
\gamma_1 &= \frac{2}{\sqrt{5}\log\big(\frac{3+\sqrt{5}}{2}\big)} = \frac{1}{\sqrt{5}\log\psi} \approx 0.9293
\end{aligned}$$

Y evaluando en $r = 3$ tenemos

$$\begin{aligned}
(\Theta(1,3))' &= \frac{\sqrt{5}}{4}\log\big(\frac{3+\sqrt{5}}{2}\big)\\
2 = \varphi(3) &= \gamma_2\frac{\sqrt{5}}{4}\log\big(\frac{3+\sqrt{5}}{2}\big)\\
\gamma_2 &= \frac{8}{\sqrt{5}\log\big(\frac{3+\sqrt{5}}{2}\big)} = \frac{4}{\sqrt{5}\log\psi} \equiv 3.7173
\end{aligned}$$

Por lo cual si $1 < r < 3$

$$\varphi(r) = \gamma_1\frac{(\beta(r))'}{(2-\beta(r))^2} \tag{98}$$

$$\Phi(x,r) = \gamma_1\frac{(\beta(r))'x^{2r} + x^r\log x - x^{3r}\log x}{(x^{2r}-\beta(r)x^r+1)^2} \tag{99}$$

Y para $3 < r < 4$

$$\varphi(r) = \gamma_2\frac{(\beta(r))'}{(2-\beta(r))^2} \tag{100}$$

$$\Phi(x,r) = \gamma_2\frac{(\beta(r))'x^{2r} + x^r\log x - x^{3r}\log x}{(x^{2r}-\beta(r)x^r+1)^2} \tag{101}$$

3.3 Caso $0 < r \leqslant 1$

Finalmente si $0 < r \leqslant 1$, definimos Θ_1

$$\Theta_1(x,r) = \frac{-x^r}{x^{2r}+\beta(r)x^r-1}$$

Y tenemos que

$$\int \Phi(x,r)dr = \gamma\Theta_1(x,r)$$

Por tanto si hacemos $x = 1$, tenemos que

$$\Theta_1(1,r) = \frac{-1}{\beta(r)}$$

Derivando obtenemos

$$(\Theta_1(1,r))' = \frac{(\beta(r))'}{\beta(r)^2}$$

Y evaluando en $r = 1$ tenemos

$$1 = \varphi(1) = \frac{\sqrt{5}\log\left(\frac{3+\sqrt{5}}{2}\right)}{2}\gamma$$
$$\gamma = \frac{2}{\sqrt{5}\log\left(\frac{3+\sqrt{5}}{2}\right)} = \frac{1}{\sqrt{5}\log\psi} \approx 0.9293$$

Con lo cual $0 < r \leqslant 1$

$$\varphi(r) = \gamma\frac{(\beta(r))'}{\beta(r)^2} \tag{102}$$
$$\Phi(x,r) = \gamma\frac{(\beta(r))'x^{2r} + x^r\log x + x^{3r}\log x}{(x^{2r} + \beta(r)x^r - 1)^2} \tag{103}$$

4 Ecuaciones diferenciales e integrales

4.1 Región 1

Sea $0 < r \leqslant 1$, $\varphi(1) = 1$, $\gamma \in \mathbb{R}$ igual a

$$\gamma = \frac{1}{\sqrt{5}\log\psi}$$

Sea $x, r \in \mathbb{R}$, $0 < r \leqslant 1$
Definimos β

$$\beta(r) = \psi^r + \tau^r = \Big(\frac{1+\sqrt{5}}{2}\Big)^r + \Big(\frac{1-\sqrt{5}}{2}\Big)^r$$

Y definimos Θ_1 como

$$\Theta_1(x, r) = \frac{-x^r}{x^{2r} + \beta(r)x^r - 1} \tag{104}$$

Teorema 41. *Sea* $x, r \in \mathbb{R}$*,* $0 < r \leqslant 1$*,* $-1 \leqslant x \leqslant 1$*,* $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Sea Φ_0 *igual a*

$$\Phi_0(x, r) = \gamma\Theta_1^2(x, r)\Big((\beta(r))' + \frac{\log x}{x^r} + x^r \log x\Big) \tag{105}$$

Entonces Φ_0 *cumple con*

$$\int \Phi_0(x, r)dr = \gamma\Theta_1(x, r) \tag{106}$$

Lema 3. *Sea* $x, r \in \mathbb{R}$*,* $0 < r \leqslant 1$*,* $-1 \leqslant x \leqslant 1$*,* $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\begin{aligned} \varphi_0(x, r) &= \gamma\frac{\Theta_1^2(x, r)}{x^{2r}}\Big((\beta(r))'x^r + \log x(x^{2r} + 1)\Big) \\ &= \gamma\frac{\Theta_1^2(x, r)}{x^{2r}}\Big(\log\big[\psi^{\psi^r x^r} * \tau^{\tau^r x^r} * x^{(x^{2r}+1)}\big]\Big) \end{aligned} \tag{107}$$

Lema 4. *Sea* $r \in \mathbb{R}$*,* $0 < r \leqslant 1$

$$\varphi_0(r) = \gamma\frac{(\beta(r))'}{\beta(r)^2} \tag{108}$$

Teorema 42. *Sea* $x, r, c_1 \in \mathbb{R}$*,* $0 < r \leqslant 1$*,* $-1 \leqslant x \leqslant 1$*,* $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\int \varphi_0(x, r)x^r dr = \gamma\Theta_1(x, r) + c_1 \tag{109}$$

Teorema 43. *Sea* $x, r \in \mathbb{R}$*,* $0 < r \leqslant 1$*,* $-1 \leqslant x \leqslant 1$*,* $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\int_0^1 -\varphi_0(x, r)x^r dr = \gamma\Big(\frac{x}{x^2 + x - 1}\Big) \tag{110}$$

4.2 Región 2 y 3

Sea $1 < r < 4$, $\varphi(2) = 1, \varphi(3) = 2$, $\gamma_1, \gamma_2 \in \mathbb{R}$ igual a

$$\gamma_1 = \frac{1}{\sqrt{5}\log\psi}, \quad \gamma_2 = \frac{4}{\sqrt{5}\log\psi}$$

Definimos β

$$\beta(r) = \psi^r + \tau^r = \Big(\frac{1+\sqrt{5}}{2}\Big)^r + \Big(\frac{1-\sqrt{5}}{2}\Big)^r \tag{111}$$

Sea $x, r \in \mathbb{R}$, $1 < r < 4$, definimos Θ como

$$\Theta(x, r) = \frac{x^r}{x^{2r} - \beta(r)x^r + 1} \tag{112}$$

Teorema 44. *Sea* $x, r \in \mathbb{R}$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Si $1 < r < 3$
Sea Φ_1 *igual a*

$$\Phi_1(x,r) = \gamma_1 \Theta^2(x,r)((\beta(r))' + \frac{\log x}{x^r} - x^r \log x) \tag{113}$$

Entonces Φ_1 *cumple con*

$$\int \Phi_1(x,r)dr = \gamma_1 \Theta(x,r) \tag{114}$$

Y si $3 \leqslant r < 4$
Sea Φ_2 *igual a*

$$\Phi_2(x,r) = \gamma_2 \Theta^2(x,r)((\beta(r))' + \frac{\log x}{x^r} - x^r \log x) \tag{115}$$

Entonces Φ_2 *cumple con*

$$\int \Phi_2(x,r)dr = \gamma_2 \Theta(x,r) \tag{116}$$

Lema 5. *Sea* $x, r \in \mathbb{R}$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$
Si $1 < r < 3$

$$\varphi_1(x,r) = \gamma_1 \frac{\Theta^2(x,r)}{x^{2r}} \Big((\beta(r))' x^r - \log x (x^{2r} - 1) \Big) \tag{117}$$

Si $3 \leqslant r < 4$,

$$\varphi_2(x,r) = \gamma_2 \frac{\Theta^2(x,r)}{x^{2r}} \Big((\beta(r))' x^r - \log x (x^{2r} - 1) \Big) \tag{118}$$

Lema 6. *Sea* $r \in \mathbb{R}$
Si $1 < r < 3$,

$$\varphi_1(r) = \gamma_1 \frac{(\beta(r))'}{(2 - \beta(r))^2} \tag{119}$$

Si $3 \leqslant r < 4$,

$$\varphi_2(r) = \gamma_2 \frac{(\beta(r))'}{(2 - \beta(r))^2} \tag{120}$$

Teorema 45. *Sea* $x, r, c_1 \in \mathbb{R}$, $1 < r < 3$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\int \varphi_1(x,r) x^r dr = \gamma_1 \Theta(x,r) + c_1 \tag{121}$$

Teorema 46. *Sea* $x, r \in \mathbb{R}$, $1 < r < 3$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\int_1^3 -\varphi_1(x,r) x^r dr = \gamma_1 \Big(\frac{x}{x^2 - x + 1} - \frac{x^3}{x^6 - 4x^3 + 1} \Big) \tag{122}$$

Teorema 47. *Sea* $x, r, c_1 \in \mathbb{R}$, $3 \leqslant r < 4$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\int \varphi_2(x,r) x^r dr = \gamma_2 \Theta(x,r) + c_1 \tag{123}$$

Teorema 48. *Sea* $x, r \in \mathbb{R}$, $3 \leqslant r < 4$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\int_3^4 -\varphi_2(x,r) x^r dr = \gamma_2 \Big(\frac{x^3}{x^6 - 4x^3 + 1} - \frac{x^4}{x^8 - 7x^4 + 1} \Big) \tag{124}$$

4.3 Región separable

Sea $r \geqslant 4$, $\varphi(4) = 3$, $\phi, \tau, \alpha, b \in \mathbb{R}$ igual a

$$\alpha = \frac{1}{\sqrt{5}}$$
$$\psi = \frac{1+\sqrt{5}}{2} \quad \tau = \frac{1-\sqrt{5}}{2}$$
$$b = \frac{\psi^4}{\log\psi} - \frac{\tau^4}{\log\tau} + \frac{3}{5}$$

Sea $x, r \in \mathbb{R}$, $r \geqslant 4$, $j = (-1)^{\frac{2}{r}} \in \mathbb{C}$, $r' = \frac{r}{2}$.
Definimos β de la siguiente manera

$$\beta(r) = \psi^r + \tau^r = \Big(\frac{1+\sqrt{5}}{2}\Big)^r + \Big(\frac{1-\sqrt{5}}{2}\Big)^r$$

Definimos Θ y Υ

$$\Theta(x,r) = \frac{\varphi_{r'}\Theta(x,r') + \varphi_{r'}\Theta(jx,r')}{\varphi_r} = \frac{x^r}{x^{2r} - \beta(r)x^r + 1}$$
$$\Upsilon(r) = (2-\beta(r))\Big(\frac{\psi^r}{\log\psi} - \frac{\tau^r}{\log\tau} - b\Big)$$

Teorema 49. *Sea $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*
Sea Φ igual a

$$\Phi_3(x,r) = \alpha\Theta^2(x,r)\Big(\Upsilon(r)((\beta(r))' + \frac{\log x}{x^r} - x^r\log x) + (\Upsilon(r))'\frac{1}{\Theta(x,r)}\Big)$$

Entonces Φ cumple con

$$\int \Phi_3(x,r)dr = \alpha\Upsilon(r)\Theta(x,r) \tag{125}$$

Lema 7. *Sea $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_3(x,r) = \alpha\frac{\Theta^2(x,r)}{x^{2r}}\Big[\Upsilon(r)\Big((\beta(r))'x^r - \log x(x^{2r}-1)\Big) + (\Upsilon(r))'\Big(x^{2r} - \beta(r)x^r + 1\Big)\Big] \tag{126}$$

Lema 8. *Sea $r \in \mathbb{R}$, $r \geqslant 4$*

$$\varphi_3(r) = \alpha\Big(\frac{(\Upsilon(r))'}{(2-\beta(r))} + \frac{(\beta(r))'\Upsilon(r)}{(2-\beta(r))^2}\Big) = \frac{(\psi^r - \tau^r)}{\sqrt{5}} \tag{127}$$

4.4 Integral A,B

Teorema 50 (Integral A)**.** *Sea $x, r, b \in \mathbb{R}$, $r \geqslant 4$, $-1 \leqslant x \leqslant 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\int \varphi(x,r)x^r dr = \alpha\Upsilon(r)\Theta(x,r) + b \tag{128}$$

Teorema 51 (Integral B)**.** *Sea $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\int_4^\infty -\varphi(x,r)x^r dr = \alpha\Big(\frac{3x^4}{x^8 - 7x^4 + 1}\Big) \tag{129}$$

5 Representaciones

5.1 Forma integral

Sea $x, r \in \mathbb{R}$, $r \geqslant 4$, definimos lo siguiente

Definición 3.

$$
\begin{aligned}
\alpha &= \sqrt{5}, \qquad \gamma = \frac{1}{\alpha}, \qquad \psi = \frac{1+\alpha}{2}, \qquad \tau = \frac{1-\alpha}{2} \\
\varphi_r &= \gamma(\psi^r - \tau^r) \\
\beta(r) &= \psi^r + \tau^r \\
\Upsilon(r) &= \gamma(2-\beta(r))\Big(\frac{\psi^r}{\log\psi} - \frac{\tau^r}{\log\tau}\Big) \\
\Theta(x,r) &= \frac{x^r}{x^{2r} - \beta(r)x^r + 1} \\
\varphi(x,r) &= \frac{\Theta^2(x,r)}{x^{2r}}\Big[\Upsilon(r)\Big((\beta(r))'x^r - \log x(x^{2r}-1)\Big) + (\Upsilon(r))'\big(x^{2r} - \beta(r)x^r + 1\big)\Big]
\end{aligned}
$$

Sea $m \in \mathbb{N}$, si $r = 2^m \geqslant 4$, entonces

Teorema 52. *Sea $k \in \mathbb{N}$, $x, y, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\frac{1}{\varphi_r}\sum_{k=0}^{\infty}\varphi_{rk}x^{rk} = \frac{1}{\Upsilon(r)}\int \varphi(x,r)x^r dr \tag{130}$$

$$\Upsilon(r)\sum_{k=0}^{\infty}\varphi_{rk}x^{rk} = \varphi_r\int \varphi(x,r)x^r dr \tag{131}$$

$$\int \varphi(x,r)x^r dr = \frac{\Upsilon(r)}{\varphi_r}\sum_{k=0}^{\infty}\varphi_{rk}x^{rk} = \int_r^{\infty} -\varphi(x,y)x^y dy \tag{132}$$

$$\sum_{k=0}^{\infty}\varphi_{rk}x^{rk} = \frac{\varphi_r}{\Upsilon(r)}\int \varphi(x,r)x^r dr \tag{133}$$

$$\varphi_r\Big(\int \varphi(x,r)x^r dr\Big) = \Big(\sum_{k=0}^{\infty}\varphi_{rk}x^{rk}\Big)\Upsilon(r) \tag{134}$$

Teorema 53. *Sea $k \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\int\int \varphi(x,r)x^r dr dx = \frac{x\Upsilon(r)}{\varphi_r}\Big(\sum_{k=0}^{\infty}\frac{\varphi_{rk}}{rk+1}x^{rk}\Big) \tag{135}$$

Teorema 54. *Sea $k, w \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_r\Big(\int \varphi(x,r)x^r dr\Big)^{w+1} = \Big(\sum_{k=0}^{\infty}\varphi_{rk}x^{rk}\Big)(\Theta(x,r))^{w}(\Upsilon(r))^{w+1} \tag{136}$$

$$\varphi_r\Big(\int \varphi(x,r)x^r dr\Big)^{-w} = \Big(\sum_{k=0}^{\infty}\varphi_{rk}x^{rk}\Big)(\Theta(x,r))^{-(w+1)}(\Upsilon(r))^{-w} \tag{137}$$

5.2 Forma con potencias

Teorema 55. *Sea* $k,z,w \in \mathbb{N}$, $x,r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^z \Big(\int \varphi(x,r) x^r dr\Big)^{z+w} = \Big(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\Big)^z (\Theta(x,r))^w (\Upsilon(r))^{z+w} \tag{138}$$

$$\varphi_r^z \Big(\int \varphi(x,r) x^r dr\Big)^{-w} = \Big(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\Big)^z (\Theta(x,r))^{-(z+w)} (\Upsilon(r))^{-w} \tag{139}$$

Teorema 56. *Sea* $k,z,w \in \mathbb{N}$, $x,r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^w \Big(\int \varphi(x,r) x^r dr\Big)^{z} = (\Upsilon(r))^z \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^w \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{-z+w} \tag{140}$$

$$\varphi_r^w \Big(\int \varphi(x,r) x^r dr\Big)^{-z} = (\Upsilon(r))^{-z} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^w \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{z+w} \tag{141}$$

$$\varphi_r^z \Big(\int \varphi(x,r) x^r dr\Big)^{z+w} = (\Upsilon(r))^{z+w} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^z \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{-w} \tag{142}$$

$$\varphi_r^z \Big(\int \varphi(x,r) x^r dr\Big)^{z-w} = (\Upsilon(r))^{z-w} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^z \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{w} \tag{143}$$

Teorema 57. *Sea* $k,z \in \mathbb{N}$, $x,r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^z \int \varphi(x,r) x^r dr = \Upsilon(r) \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^z \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{z-1} \tag{144}$$

$$\frac{1}{\varphi_r^z} \int \varphi(x,r) x^r dr = \Upsilon(r) \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{-z} \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{-z-1} \tag{145}$$

$$\varphi_r \Big(\int \varphi(x,r) x^r dr\Big)^{z} = (\Upsilon(r))^z \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right) \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{-z+1} \tag{146}$$

$$\varphi_r \Big(\int \varphi(x,r) x^r dr\Big)^{-z} = (\Upsilon(r))^{-z} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right) \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{z+1} \tag{147}$$

$$\varphi_r^{z+1} \Big(\int \varphi(x,r) x^r dr\Big)^{z} = (\Upsilon(r))^z \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{z+1} \Big(x^r + \frac{1}{x^r} - \beta(r)\Big) \tag{148}$$

$$\frac{1}{\varphi_r^{z-1}} \Big(\int \varphi(x,r) x^r dr\Big)^{-z} = (\Upsilon(r))^{-z} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{-z+1} \Big(x^r + \frac{1}{x^r} - \beta(r)\Big) \tag{149}$$

Lema 9. *Sea* $k \in \mathbb{N}$, $x,r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^2 \int \varphi(x,r) x^r dr = \Upsilon(r) \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^2 \Big(x^r + \frac{1}{x^r} - \beta(r)\Big) \tag{150}$$

$$\varphi_r \Big(\int \varphi(x,r) x^r dr\Big)^{-1} = \frac{1}{\Upsilon(r)} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right) \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^2 \tag{151}$$

5.3 Forma diferencial

Teorema 58. *Sea* $k \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)' = \left(\sum_{k=0}^{\infty} \frac{(\varphi_{rk})'\varphi_r - \varphi_{rk}(\varphi_r)'}{(\varphi_r)^2} x^{rk} + (k \log x) \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right) \tag{152}$$

$$\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)' = \left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^2 \Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big) \tag{153}$$

Definición 4.

$$\Xi(x, r) = \left[(\Upsilon(r))'\Big(x^r + \frac{1}{x^r} - \beta(r)\Big) + \Upsilon(r)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big)\right] \tag{154}$$

Teorema 59. *Sea* $k \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi(x, r) = \frac{1}{x^r} \left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^2 \Xi(x, r) \tag{155}$$

Teorema 60. *Sea* $k, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^{2z}\, \varphi(x, r)^z = \frac{1}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{2z} \Xi(x, r)^z \tag{156}$$

Teorema 61. *Sea* $k, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^{2z}\, \varphi(x, r)^z \Big(\int \varphi(x, r) x^r dr\Big)^z = \frac{(\Upsilon(r))^z}{x^{rz}} \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{-z} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{2z} \Xi(x, r)^z \tag{157}$$

Teorema 62. *Sea* $k, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^{3z}\, \varphi(x, r)^z \Big(\int \varphi(x, r) x^r dr\Big)^z = \frac{(\Upsilon(r))^z}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{3z} \Xi(x, r)^z \tag{158}$$

Teorema 63. *Sea* $k, w, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi(x, r)^z \Big(\int \varphi(x, r) x^r dr\Big)^w = \frac{(\Upsilon(r))^w}{x^{rz}} \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{-2z-w} \Xi(x, r)^z \tag{159}$$

Teorema 64. *Sea* $k, w, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^{w}\, \varphi(x, r)^z \Big(\int \varphi(x, r) x^r dr\Big)^w = \frac{(\Upsilon(r))^w}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{w} \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{-2z} \Xi(x, r)^z \tag{160}$$

Teorema 65. *Sea* $k, w, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^{2z}\, \varphi(x, r)^z \Big(\int \varphi(x, r) x^r dr\Big)^w = \frac{(\Upsilon(r))^w}{x^{rz}} \Big(x^r + \frac{1}{x^r} - \beta(r)\Big)^{-w} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{2z} \Xi(x, r)^z \tag{161}$$

Teorema 66. *Sea* $k, w, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^{2z+w}\, \varphi(x, r)^z \Big(\int \varphi(x, r) x^r dr\Big)^w = \frac{(\Upsilon(r))^w}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{2z+w} \Xi(x, r)^z \tag{162}$$

Definición 5.

$$\chi(x,r) = (\Xi(x,r))' + 2\left(\sum_{k=0}^{\infty}\frac{\varphi_{rk}}{\varphi_r}x^{rk}\right)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r\log x\Big)\Xi(x,r) \tag{163}$$

Teorema 67. *Sea $k\in\mathbb{N}$, $x,r\in\mathbb{R}$, $r\geqslant 4$, $-1<x<1$, $x\neq\{0,\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$*

$$(\varphi(x,r))' = \frac{1}{x^r}\left(\sum_{k=0}^{\infty}\frac{\varphi_{rk}}{\varphi_r}x^{rk}\right)^2\chi(x,r) - (\log x)\,\varphi(x,r) \tag{164}$$

$$= \frac{1}{x^r}\left(\sum_{k=0}^{\infty}\frac{\varphi_{rk}}{\varphi_r}x^{rk}\right)^2\Big[\chi(x,r) - (\log x)\,\Xi(x,r)\Big] \tag{165}$$

Teorema 68. *Sea $k,z\in\mathbb{N}$, $x,r\in\mathbb{R}$, $r\geqslant 4$, $-1<x<1$, $x\neq\{0,\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_r^{2z}\left((\varphi(x,r))' + (\log x)\varphi(x,r)\right)^z = \frac{1}{x^{rz}}\left(\sum_{k=0}^{\infty}\varphi_{rk}x^{rk}\right)^{2z}\chi(x,r)^z \tag{166}$$

Teorema 69. *Sea $k,w,z\in\mathbb{N}$, $x,r\in\mathbb{R}$, $r\geqslant 4$, $-1<x<1$, $x\neq\{0,\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_r^{2z+w}\left((\varphi(x,r))' + (\log x)\varphi(x,r)\right)^z\left(\int\varphi(x,r)x^r dr\right)^w = \frac{(\Upsilon(r))^w}{x^{rz}}\left(\sum_{k=0}^{\infty}\varphi_{rk}x^{rk}\right)^{2z+w}\chi(x,r)^z \tag{167}$$

Teorema 70. *Sea $k,z\in\mathbb{N}$, $x,r\in\mathbb{R}$, $r\geqslant 4$, $-1<x<1$, $x\neq\{0,\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_r^{3z}\left((\varphi(x,r))' + (\log x)\varphi(x,r)\right)^z\left(\int\varphi(x,r)x^r dr\right)^z = \frac{(\Upsilon(r))^z}{x^{rz}}\left(\sum_{k=0}^{\infty}\varphi_{rk}x^{rk}\right)^{3z}\chi(x,r)^z \tag{168}$$

Teorema 71. *Sea $k,z\in\mathbb{N}$, $x,r\in\mathbb{R}$, $r\geqslant 4$, $-1<x<1$, $x\neq\{0,\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_r^{5z}\,\varphi(x,r)^z\left((\varphi(x,r))' + (\log x)\varphi(x,r)\right)^z\left(\int\varphi(x,r)x^r dr\right)^z = \frac{(\Upsilon(r))^z}{x^{2rz}}\left(\sum_{k=0}^{\infty}\varphi_{rk}x^{rk}\right)^{5z}\Xi(x,r)^z\,\chi(x,r)^z \tag{169}$$

Teorema 72. *Sea $k,u,v,w,z\in\mathbb{N}$, $x,r\in\mathbb{R}$, $r\geqslant 4$, $-1<x<1$, $x\neq\{0,\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$,*
$z=2(u+v)+w$

$$\varphi_r^{z}\,\varphi(x,r)^u\left((\varphi(x,r))' + (\log x)\varphi(x,r)\right)^v\left(\int\varphi(x,r)x^r dr\right)^w = \frac{\Upsilon(r)^w\,\Xi(x,r)^u\,\chi(x,r)^v}{x^{r(u+v)}}\left(\sum_{k=0}^{\infty}\varphi_{rk}x^{rk}\right)^{z} \tag{170}$$

Definición 6.

$$\chi_2(x,r) = (\chi(x,r))' + 2\left(\sum_{k=0}^{\infty}\frac{\varphi_{rk}}{\varphi_r}x^{rk}\right)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r\log x\Big)\chi(x,r) \tag{171}$$

Teorema 73. *Sea $k\in\mathbb{N}$, $x,r\in\mathbb{R}$, $r\geqslant 4$, $-1<x<1$, $x\neq\{0,\frac{1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$*

$$(\varphi(x,r))'' = \frac{1}{x^r}\left(\sum_{k=0}^{\infty}\frac{\varphi_{rk}}{\varphi_r}x^{rk}\right)^2\chi_2(x,r) - 2\,(\log x)(\varphi(x,r))' - (\log x)^2\varphi(x,r) \tag{172}$$

$$= \frac{1}{x^r}\left(\sum_{k=0}^{\infty}\frac{\varphi_{rk}}{\varphi_r}x^{rk}\right)^2\Big[\chi_2(x,r) - 2\,(\log x)\,\chi(x,r) + (\log x)^2\,\Xi(x,r)\Big] \tag{173}$$

Teorema 74. *Sea* $k, u, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $u = 2z$

$$\varphi_r^u \Big((\varphi(x,r))'' + 2\,(\log x)(\varphi(x,r))' + (\log x)^2 \varphi(x,r)\Big)^z = \frac{\chi_2(x,r)^z}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^u \tag{174}$$

Teorema 75. *Sea* $k, u, w, z \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $u = 2z + w$

$$\varphi_r^u \Big((\varphi(x,r))'' + 2\,(\log x)(\varphi(x,r))' + (\log x)^2 \varphi(x,r)\Big)^z \Big(\int \varphi(x,r) x^r dr\Big)^w = \frac{\Upsilon(r)^w\, \chi_2(x,r)^z}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^u \tag{175}$$

Definición 7.

$$\chi_3(x,r) = (\chi_2(x,r))' + 2\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big)\chi_2(x,r) \tag{176}$$

Teorema 76. *Sea* $k \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$(\varphi(x,r))''' = \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^2 \chi_3(x,r) - 3\,(\log x)(\varphi(x,r))'' - 3(\log x)^2(\varphi(x,r))' - (\log x)^3 \varphi(x,r) \tag{177}$$

$$= \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^2 \Big[\chi_3(x,r) - 3\,(\log x)\,\chi_2(x,r) + 3\,(\log x)^2\,\chi(x,r) - (\log x)^3\,\Xi(x,r)\Big] \tag{178}$$

5.4 N-ésimas derivadas

Definición 8.

$$\chi_0 = \Xi, \quad \chi_1 = \chi, \tag{179}$$

$$\chi_{n+1}(x,r) = (\chi_n(x,r))' + 2\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big)\chi_n(x,r) \tag{180}$$

Teorema 77. *Sea* $k, n, \tau_k^n \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi(x,r)^{(n)} = \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^2 \chi_n(x,r) - \sum_{k=0}^{n-1} \tau_k^n\,(\log x)^{n-k}\,\varphi(x,r)^{(k)} \tag{181}$$

$$= \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^2 \left[\sum_{k=0}^{n} (-1)^{n-k}\,\tau_k^n\,(\log x)^{n-k}\,\chi_k(x,r)\right] \tag{182}$$

Teorema 78. *Sea* $k, n, u, z, \tau_k^n \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $u = 2z$

$$\varphi_r^u \left[\sum_{k=0}^{n} \tau_k^n\,(\log x)^{n-k}\,\varphi(x,r)^{(k)}\right]^z = \frac{\chi_n(x,r)^z}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^u \tag{183}$$

Teorema 79. *Sea* $k, n, u, w, z, \tau_k^n \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $u = 2z + w$

$$\varphi_r^u \left[\sum_{k=0}^{n} \tau_k^n\,(\log x)^{n-k}\,\varphi(x,r)^{(k)}\right]^z \Big(\int \varphi(x,r) x^r dr\Big)^w = \frac{\Upsilon(r)^w\, \chi_n(x,r)^z}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^u \tag{184}$$

Teorema 80. *Sea* $j, k, n, s, v_j, w, z, \tau_k^j \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $s = \sum v_k$, $\quad z = 2s + w$

$$\varphi_r^z \left[\prod_{j=0}^{n} \left(\sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi(x,r)^{(k)} \right)^{v_j} \right] \Big(\int \varphi(x,r) x^r dr \Big)^w = \frac{\Upsilon(r)^w \, \prod_{j=0}^{n} \chi_j(x,r)^{v_j}}{x^{rs}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^z \tag{185}$$

Lema 10. *Sea* $j, k, n, w, z, \tau_k^j \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $z = 2n + w$

$$\varphi_r^z \left[\prod_{j=0}^{n-1} \sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi(x,r)^{(k)} \right] \Big(\int \varphi(x,r) x^r dr \Big)^w = \frac{\Upsilon(r)^w \, \prod_{j=0}^{n-1} \chi_j(x,r)}{x^{rn}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^z \tag{186}$$

Lema 11. *Sea* $k, n \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geqslant 4$

$$\varphi_r \left[\prod_{k=1}^{n-1} \varphi_r^{(k)} \right] \Big(2 - \beta(r) \Big)^{2n} = \left[\prod_{k=0}^{n-1} \chi_k(1,r) \right] \tag{187}$$

Lema 12. *Sea* $k, n \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geqslant 4$

$$\varphi_r \left[\prod_{k=1}^{n-1} \varphi_r^{(k)} \right] \Big(\int \varphi_r \, dr \Big) \Big(2 - \beta(r) \Big)^{2n+1} = \Upsilon(r) \left[\prod_{k=0}^{n-1} \chi_k(1,r) \right] \tag{188}$$

Teorema 81 (Representación A)**.** *Sea* $j, k, n, s, t, u_k, v, v_j, w, z, \tau_k^j \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$, $\quad t = \sum u_k$, $\quad s = \sum v_k$, $\quad z = 2s + w + u_0$

$$\begin{aligned} \varphi_r^z \left[\prod_{k=1}^{n} \Big(\varphi_r^{(k)} \Big)^{u_k} \right] \Big(\int \varphi_r \, dr \Big)^v \left[\prod_{j=0}^{n} \left(\sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi(x,r)^{(k)} \right)^{v_j} \right] \Big(\int \varphi(x,r) x^r dr \Big)^w \Big(2 - \beta(r) \Big)^{2t+v} = \\ \Upsilon(r)^{v+w} \left[\prod_{k=0}^{n} \Big(\chi_k(1,r) \Big)^{u_k} \right] \left[\prod_{j=0}^{n} \Big(\chi_j(x,r) \Big)^{v_j} \right] \Big(\frac{1}{x^{rs}} \Big) \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{2s+w} \end{aligned} \tag{189}$$

Demostración. Tenemos que podemos expresar la serie infinita de la siguiente manera

$$\begin{aligned} \sum_{k=0}^{\infty} \varphi_{rk} x^{rk} &= \frac{\varphi_r x^r}{x^{2r} - \beta_r x^r + 1} = \varphi_r \, \Theta(x,r) \\ \sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk} &= \Theta(x,r) \end{aligned}$$

Y dado que la siguiente integral se descompone en Θ

$$\int \varphi(x,r) x^r dr = \Upsilon(r) \, \Theta(x,r)$$

podemos factorizar φ_r de la siguiente manera

$$\begin{aligned} \int \varphi(x,r) x^r dr &= \Upsilon(r) \sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk} \\ &= \frac{\Upsilon(r)}{\varphi_r} \sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \end{aligned}$$

y sabemos por la construcción de Υ que φ_r no lo divide

$$\Upsilon(r) = \gamma(2 - \beta(r)) \Big(\frac{\psi^r}{\log \psi} - \frac{\tau^r}{\log \tau} \Big)$$

Por lo cual la integral no aumenta a la potencia w

$$\varphi_r^w \left(\int \varphi(x,r) x^r dr \right)^w = \Upsilon(r)^w \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^w$$

Para $\varphi(x,r)$ en cambio es distinto

$$\varphi_r^2 \, \varphi(x,r) = \frac{1}{x^r} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^2 \Xi(x,r),$$

se aumenta en uno más a la potencia, generando dos a lo más ya que Ξ contiene un sumando que multiplica la derivada del recíproco de Θ

$$\Xi(x,r) = \left[(\Upsilon(r))' \Big(x^r + \frac{1}{x^r} - \beta(r) \Big) + \Upsilon(r) \Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x \Big) \right]$$

y ya que tenemos lo siguiente

$$\left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)' = \left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk} \right)^2 \Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x \Big)$$

podemos deducir que

$$\begin{aligned} \left(\left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{-1} \right)' &= - \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{-2} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)' \\ &= - \Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x \Big) \end{aligned}$$

por tanto φ_r no es factor de Ξ, y en general para sus derivadas pasa lo mismo

$$\varphi_r^2 \, \varphi(x,r)^{(n)} = \frac{1}{x^r} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^2 \left[\sum_{k=0}^{n} (-1)^{n-k} \, \tau_k^n \, (\log x)^{n-k} \, \chi_k(x,r) \right]$$

Por lo cual, la suma de las derivadas de $\varphi(x,r)$ aporta dos a la potencia y el producto de estas sumas genera la suma de sus potencias por dos, sea $s = \sum v_k$

$$\varphi_r^{2s} \left[\prod_{j=0}^{n} \left(\sum_{k=0}^{j} \tau_k^j \, (\log x)^{j-k} \, \varphi(x,r)^{(k)} \right)^{v_j} \right] = \left[\prod_{j=0}^{n} \Big(\chi_j(x,r) \Big)^{v_j} \right] \left(\frac{1}{x^{rs}} \right) \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{2s}$$

Para el caso de las derivadas de φ_r, aquí no se puede factorizar φ_r ya que tendrían que diferir en una constante

$$\begin{aligned} \varphi_r &= \alpha \Big(\psi^r - \tau^r \Big) \\ \varphi_r^{(n)} &= \alpha \Big(\psi^r (\log \psi)^n - \tau^r (\log \tau)^n \Big) \end{aligned}$$

Y para el caso de la integral de φ_r pasa lo mismo

$$\int \varphi_r \, dr = \alpha \Big(\frac{\psi^r}{\log \psi} - \frac{\tau^r}{\log \tau} \Big)$$

Por lo cual el producto de φ_r con sus derivadas y su integral factoriza una única φ_r, sea $t = \sum u_k$

$$\varphi_r \left[\prod_{k=1}^{n} \Big(\varphi_r^{(k)} \Big)^{u_k} \right] \Big(\int \varphi_r \, dr \Big)^v = \Upsilon(r)^v \left[\prod_{k=0}^{n} \Big(\chi_k(1,r) \Big)^{u_k} \right] \Big(\frac{1}{2 - \beta(r)} \Big)^{2t+v}$$

■

5.5 Series diferenciales

Sea $\psi = \frac{1+\sqrt{5}}{2}$, $\tau = \frac{1-\sqrt{5}}{2}$, definimos β de la siguiente manera

$$\beta(r) = \psi^r + \tau^r = \Big(\frac{1+\sqrt{5}}{2}\Big)^r + \Big(\frac{1-\sqrt{5}}{2}\Big)^r$$

Y definimos Θ de la siguiente manera

$$\Theta(x,r) = \frac{x^r}{x^{2r} - \beta(r)x^r + 1}$$

Y sea Υ_n una función derivable en r y $\alpha, b \in \mathbb{R}$, entonces podemos definir Ω de la siguiente manera

$$\Omega_n(x,r) = \int \Phi_n(x,r)dr = \Upsilon_n(r)\,\Theta^n(x,r) + b$$

Derivando β con respecto a r obtenemos

$$(\beta(r))' = \psi^r \log\psi + \tau^r \log\tau$$

Derivando Θ con respecto a r obtenemos

$$\begin{aligned}\Big[\Theta^n(x,r)\Big]' = \left[\left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^n\right]' &= n\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^{n+1} \Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big) \\ &= n\,\Theta(x,r)^{n+1}\Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big)\end{aligned}$$

Por tanto

$$\Phi_n(x,r) = \Theta^{n+1}(x,r)\Big(n\,\Upsilon_n(r)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r\log x\Big) + (\Upsilon_n(r))'\Big(x^r + \frac{1}{x^r} - \beta(r)\Big)\Big) \tag{190}$$

Y además

$$\Upsilon_n(r) = \gamma\Big(2 - \beta(r)\Big)^n\Big(\frac{\psi^r}{\log\psi} - \frac{\tau^r}{\log\tau} - b\Big), \qquad \gamma = \frac{1}{\sqrt{5}} \tag{191}$$

Entonces si hacemos

$$\Xi(x,r,n) = n\,\Upsilon_n(r)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r\log x\Big) + (\Upsilon_n(r))'\Big(x^r + \frac{1}{x^r} - \beta(r)\Big)$$

Tenemos que

$$\varphi_n(x,r) = \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^{n+1} \Xi(x,r,n)$$

Y si hacemos

$$\chi(x,r,n) = (\Xi(x,r,n))' + (n+1)\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r\log x\Big)\Xi(x,r,n)$$

Tenemos que

$$\begin{aligned}(\varphi_n(x,r))' &= \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^{n+1} \chi(x,r,n) - (\log x)\,\varphi_n(x,r) \\ &= \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^{n+1} \Big[\chi(x,r,n) - (\log x)\,\Xi(x,r,n)\Big]\end{aligned}$$

Definición 9.

$$\chi_0 = \Xi, \quad \chi_1 = \chi, \tag{192}$$

$$\chi_{j+1}(x,r,n) = (\chi_j(x,r,n))' + (n+1)\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)\Big((\beta(r))' + \frac{\log x}{x^r} - x^r \log x\Big)\chi_j(x,r,n) \tag{193}$$

Teorema 82. *Sea* $j,k,n,\tau_k^j \in \mathbb{N},\ x,r \in \mathbb{R},\ r \geqslant 4,\ -1<x<1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_n(x,r)^{(j)} = \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^{n+1} \chi_j(x,r,n) - \sum_{k=0}^{j-1} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)} \tag{194}$$

$$= \frac{1}{x^r}\left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk}\right)^{n+1} \left[\sum_{k=0}^{j} (-1)^{j-k}\, \tau_k^j \,(\log x)^{j-k}\, \chi_k(x,r,n)\right] \tag{195}$$

Teorema 83. *Sea* $j,k,n,u,z,\tau_k^j \in \mathbb{N},\ x,r \in \mathbb{R},\ r \geqslant 4,\ -1<x<1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\},$
$u = (n+1)z$

$$\varphi_r^u \left[\sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)}\right]^z = \frac{\chi_j(x,r,n)^z}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^u \tag{196}$$

Teorema 84. *Sea* $j,k,n,u,w,z,\tau_k^j \in \mathbb{N},\ x,r \in \mathbb{R},\ r \geqslant 4,\ -1<x<1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\},$
$u = (n+1)z + nw$

$$\varphi_r^u \left[\sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)}\right]^z \Big(\int \varphi_n(x,r) x^r dr\Big)^w = \frac{\Upsilon_n(r)^w\, \chi_j(x,r,n)^z}{x^{rz}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^u \tag{197}$$

Teorema 85. *Sea* $j,k,n,s,t,v_j,w,z,\tau_k^j \in \mathbb{N},\ x,r \in \mathbb{R},\ r \geqslant 4,\ -1<x<1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\},$
$s = \sum v_k, \quad z = (n+1)s + nw$

$$\varphi_r^z \left[\prod_{j=0}^{t} \sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)}\right]^{v_j} \Big(\int \varphi_n(x,r) x^r dr\Big)^w = \frac{\Upsilon_n(r)^w\, \prod_{j=0}^{t} \chi_j(x,r,n)^{v_j}}{x^{rs}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^z \tag{198}$$

Lema 13. *Sea* $j,k,n \in \mathbb{N},\ r \in \mathbb{R},\ r \geqslant 4$

$$\varphi_r \prod_{k=1}^{j-1} \varphi_r^{(k)} \Big(2 - \beta(r)\Big)^{(n+1)j} = \left[\prod_{k=0}^{j-1} \chi_k(1,r,n)\right] \tag{199}$$

Lema 14. *Sea* $j,k,n \in \mathbb{N},\ r \in \mathbb{R},\ r \geqslant 4$

$$\varphi_r \left[\prod_{k=1}^{j-1} \varphi_r^{(k)}\right] \Big(\int \varphi_r\, dr\Big) \Big(2 - \beta(r)\Big)^{(n+1)j+n} = \Upsilon(r) \left[\prod_{k=0}^{j-1} \chi_k(1,r,n)\right] \tag{200}$$

Teorema 86 (Representación B)**.** *Sea* $j,k,n,s,t,u_k,v,v_j,w,z,\tau_k^j \in \mathbb{N},\ x,r \in \mathbb{R},\ r \geqslant 4,\ -1<x<1,$
$x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}, \quad s = \sum v_k, \quad t = \sum u_k, \quad z = (n+1)s + nw + u_0$

$$\varphi_r^z \left[\prod_{k=1}^{i} \Big(\varphi_r^{(k)}\Big)^{u_k}\right] \Big(\int \varphi_r\, dr\Big)^v \left[\prod_{j=0}^{i} \left(\sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)}\right)^{v_j}\right] \Big(\int \varphi_n(x,r) x^r dr\Big)^w \Big(2 - \beta(r)\Big)^{(n+1)t+nv} =$$

$$\Upsilon(r)^v\, \Upsilon_n(r)^w \left[\prod_{k=0}^{i} \Big(\chi_k(1,r,n)\Big)^{u_k}\right] \left[\prod_{j=0}^{i} \Big(\chi_j(x,r,n)\Big)^{v_j}\right] \left(\frac{1}{x^{rs}}\right) \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{(n+1)s+nw} \tag{201}$$

5.6 Forma entre cocientes

Teorema 87. *Sea* $j, k, n, \tau_k^j \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r \left(\sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)} \right) \left(\int \varphi_n(x,r) x^r dr \right)^{-1} = \frac{\chi_j(x,r,n)}{x^r\, \Upsilon_n(r)} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right) \tag{202}$$

Teorema 88. *Sea* $j, k, n, s, \tau_k^j \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\prod_{j=0}^{s-1} \sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)} \right] \left(\int \varphi_n(x,r) x^r dr \right)^{-s} = \frac{\prod_{j=0}^{s-1} \chi_j(x,r,n)}{x^{rs}\, \Upsilon_n(r)^s} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^s \tag{203}$$

Corolario 16. *Sea* $j, n \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geqslant 4$

$$\varphi_r^{(j)} \left(\int \varphi_r \, dr \right)^{-1} \Big(2 - \beta(r) \Big) = \frac{\chi_j(1,r,n)}{\Upsilon_n(r)} \tag{204}$$

Lema 15. *Sea* $j, n, s \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geqslant 4$

$$\varphi_r \prod_{j=1}^{s-1} \varphi_r^{(j)} \left(\int \varphi_r \, dr \right)^{-s} \Big(2 - \beta(r) \Big)^s = \frac{\prod_{j=0}^{s-1} \chi_j(1,r,n)}{\Upsilon_n(r)^s} \tag{205}$$

Definición 10.

$$\rho(x,r,n,s) = \left[\frac{1}{\prod_{j=1}^{s} \varphi_r^{(j)}} \right] \prod_{j=1}^{s} \sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)} \left(\int \varphi_n(x,r) x^r dr \right)^{-s} \tag{206}$$

$$\tau(x,r,n,s) = \left[\frac{1}{\prod_{j=1}^{s} \chi_j(1,r,n)} \right] \frac{\prod_{j=1}^{s} \chi_j(x,r,n)}{x^{rs}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^s \tag{207}$$

Teorema 89 (Representación C)**.** *Sea* $n, s \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{1}{2 - \beta(r)} \right]^s \left(\int \varphi_r \, dr \right)^s \rho(x,r,n,s) = \tau(x,r,n,s) \tag{208}$$

Lema 16. *Sea* $r \in \mathbb{N}$, $r \geqslant 4$

$$\frac{\varphi_r}{2 - \beta(r)} = \frac{\varphi_r^2}{2\varphi_r - \varphi_{2r}} \tag{209}$$

Teorema 90 (Representación D)**.** *Sea* $n, s \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{\varphi_r}{2\varphi_r - \varphi_{2r}} \right]^s \left(\int \varphi_r \, dr \right)^s \rho(x,r,n,s) = \tau(x,r,n,s) \tag{210}$$

Lema 17. *Sea* $j, k, n \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\int \varphi(x,r) x^r dr \int \varphi_n(x,r) x^r dr = \frac{\Upsilon(r)\, \Upsilon_n(r)\, x^r \, \sum_{k=0}^{j} \tau_k^j \,(\log x)^{j-k}\, \varphi_n(x,r)^{(k)}}{\chi_j(x,r,n)} \tag{211}$$

Corolario 17. *Sea* $n \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geqslant 4$

$$\varphi_r \left(\int \varphi_r \, dr \right)^{-2} = \frac{\Xi(1,r,n)}{\Upsilon(r)\, \Upsilon_n(r)} \tag{212}$$

Lema 18. *Sea* $j, n \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geqslant 4$

$$\varphi_r^{(j)} \left(\int \varphi_r \, dr \right)^{-2} = \frac{\chi_j(1,r,n)}{\Upsilon(r)\, \Upsilon_n(r)} \tag{213}$$

Lema 19. *Sea* $j, n, s \in \mathbb{N}$, $r \in \mathbb{R}$, $r \geqslant 4$

$$\varphi_r \prod_{j=1}^{s-1} \varphi_r^{(j)} \left(\int \varphi_r \, dr \right)^{-2s} = \frac{\prod_{j=0}^{s-1} \chi_j(1, r, n)}{\Upsilon(r)^s \, \Upsilon_n(r)^s} \tag{214}$$

Teorema 91 (Representación E)**.** *Sea* $n, s \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{1}{\Upsilon(r)} \right]^s \left(\int \varphi_r \, dr \right)^{2s} \rho(x, r, n, s) = \tau(x, r, n, s) \tag{215}$$

Teorema 92 (Representación F)**.** *Sea* $n, s, u \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{(2 - \beta(r))^{u-1}}{\Upsilon(r)^u} \right]^s \left(\int \varphi_r \, dr \right)^{(u+1)s} \rho(x, r, n, s) = \tau(x, r, n, s) \tag{216}$$

Definición 11. *Sea* β *una función derivable en* r*, definimos* Γ *como*

$$\Gamma(r) = (2 + \beta(r)) \left[\frac{\psi^r}{\log \psi} + \frac{\tau^r}{\log \tau} \right] \tag{217}$$

Lema 20. *Sea* $r \in \mathbb{R}$, $r \geqslant 4$

$$\Upsilon'(2r) = \Upsilon(r) \, \Gamma(r) = 4 * (2 - \beta(2r)) \int \int \varphi_{2r} \, dr \, dr \tag{218}$$

Teorema 93 (Representación G)**.** *Sea* $s, u \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{(2 - \beta(r))^{u-1} \, \Gamma(r)^u}{\Upsilon'(2r)^u} \right]^s \left(\int \varphi_r \, dr \right)^{(u+1)s} \rho(x, r, 1, s) = \tau(x, r, 1, s) \tag{219}$$

Lema 21. *Sea* $r \in \mathbb{R}$, $r \geqslant 4$

$$\Gamma(r) = (2 + \beta(r)) \int \beta_r \, dr \tag{220}$$

Teorema 94 (Representación H)**.** *Sea* $s, u, v \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{(2 - \beta(r))^{u-1} \, (2 + \beta(r))^{v+1}}{\Upsilon(r)^u \, \Gamma(r)^{v+1}} \right]^s \left(\int \varphi_r \, dr \right)^{(u+1)s} \left(\int \beta_r \, dr \right)^{(v+1)s} \rho(x, r, 1, s) = \tau(x, r, 1, s) \tag{221}$$

Lema 22. *Sea* $r \in \mathbb{R}$, $r \geqslant 4$

$$\int \beta_r \, dr \int \varphi_r \, dr = 4 \int \int \varphi_{2r} \, dr \, dr \tag{222}$$

Teorema 95 (Representación I)**.** *Sea* $s, u \in \mathbb{N}$, $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s \left[\frac{4^{u+1} \, (2 - \beta(r))^{u-1} \, (2 + \beta(r))^{u+1}}{\Upsilon(r)^u \, \Gamma(r)^{u+1}} \right]^s \left(\int \int \varphi_{2r} \, dr \, dr \right)^{(u+1)s} \rho(x, r, 1, s) = \tau(x, r, 1, s) \tag{223}$$

6 Formas canónicas

6.1 Fórmulas canónicas

Teorema 96. *Sea* $j, k, n, s, \tau_k^j \in \mathbb{N},\ x, r \in \mathbb{R},\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^s = \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^s \left[\frac{\prod_{j=0}^{s-1} \chi_j(x,r,n)\ \left(\int \varphi_n(x,r) x^r\, dr\right)^s}{x^{rs} \Upsilon_n(r)^s\ \prod_{j=0}^{s-1} \sum_{k=0}^{j} \tau_k^j\, (\log x)^{j-k}\, \varphi_n(x,r)^{(k)}}\right] \tag{224}$$

Teorema 97 (Fórmula 1)**.** *Sea* $j, k, n, \tau_k^j \in \mathbb{N},\ x, r, z \in \mathbb{R},\ z \geqslant 1,\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\varphi_r^z = \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^z \left[\frac{\chi_j(x,r,n)\ \left(\int \varphi_n(x,r) x^r\, dr\right)}{x^r\ \Upsilon_n(r)\ \sum_{k=0}^{j} \tau_k^j\, (\log x)^{j-k}\, \varphi_n(x,r)^{(k)}}\right]^z \tag{225}$$

Definición 12.

$$\sigma(r,n,j) = \frac{\Upsilon_n(r)\, \varphi_r^{(j)}}{\chi_j(1,r,n)} \tag{226}$$

Teorema 98. *Sea* $j, n, u, v \in \mathbb{N},\ r, z \in \mathbb{R},\ r \geqslant 4$

$$\left[\frac{1}{2-\beta(r)}\right]^z = \sigma(r,n,j)^z \left(\int \varphi_r\, dr\right)^{-z} \tag{227}$$

$$\left[\frac{(2-\beta(r))^{u-1}}{\Upsilon(r)^u}\right]^z = \sigma(r,n,j)^z \left(\int \varphi_r\, dr\right)^{-(u+1)z} \tag{228}$$

$$\left[\frac{(2-\beta(r))^{u-1}\, \Gamma(r)^u}{\Upsilon'(2r)^u}\right]^z = \sigma(r,1,j)^z \left(\int \varphi_r\, dr\right)^{-(u+1)z} \tag{229}$$

$$\left[\frac{(2-\beta(r))^{u-1}\, (2+\beta(r))^{v+1}}{\Upsilon(r)^u\, \Gamma(r)^{v+1}}\right]^z = \sigma(r,1,j)^z \left(\int \varphi_r\, dr\right)^{-(u+1)z} \left(\int \beta_r\, dr\right)^{-(v+1)z} \tag{230}$$

$$\left[\frac{4^{u+1}\, (2-\beta(r))^{u-1}\, (2+\beta(r))^{u+1}}{\Upsilon(r)^u\, \Gamma(r)^{u+1}}\right]^z = \sigma(r,1,j)^z \left(\int\int \varphi_{2r}\, dr\, dr\right)^{-(u+1)z} \tag{231}$$

Teorema 99. *Sea* $u \in \mathbb{N},\ r, z \in \mathbb{R},\ r \geqslant 4$

$$\left[\frac{1}{2-\beta(r)}\right]^z = \left[\frac{(2-\beta(r))^{u-1}}{\Upsilon(r)^u}\right]^z \left(\int \varphi_r\, dr\right)^{uz} \tag{232}$$

$$\left[\frac{1}{2-\beta(r)}\right]^z = \left[\frac{4^u (2-\beta(r))^{u-1}\, (2+\beta(r))^u}{\Upsilon(r)^u\, \Gamma(r)^u}\right]^z \left(\int\int \varphi_2 r\, dr\, dr\right)^{uz} \tag{233}$$

$$\left[\frac{1}{2-\beta(r)}\right]^z = \left[\frac{4^{u+1}\, (2-\beta(r))^{u-1}\, (2+\beta(r))^{u+1}}{\Upsilon(r)^u\, \Gamma(r)^{u+1}}\right]^z \left(\int \varphi_r\, dr\right)^{-z} \left(\int\int \varphi_{2r}\, dr\, dr\right)^{(u+1)z} \tag{234}$$

Teorema 100. *Sea* $u, v \in \mathbb{N},\ r, z \in \mathbb{R},\ r \geqslant 4$

$$\left[\frac{1}{2-\beta(r)}\right]^z = \left[\frac{(2-\beta(r))^{u-1}\, (2+\beta(r))^v}{\Upsilon(r)^u\, \Gamma(r)^v}\right]^z \left(\int \varphi_r\, dr\right)^{uz} \left(\int \beta_r\, dr\right)^{vz} \tag{235}$$

Teorema 101. *Sea* $j_1, j_2, k, u, v, \tau_k^j \in \mathbb{N},\ x, r, z \in \mathbb{R},\ z \geqslant 1,\ r \geqslant 4,\ -1 < x < 1,\ x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$

$$\begin{gathered}\varphi_r^z \left[\frac{(2-\beta(r))^{u-1}\, (2+\beta(r))^v}{\Upsilon(r)^u\, \Gamma(r)^v}\right]^z \left(\int \varphi_r\, dr\right)^{(u+1)z} \left(\int \beta_r\, dr\right)^{vz} = \\ \left[\frac{\varphi_r^{(j_2)}}{\chi_{j_2}(1,r)}\right]^z \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^z \left[\frac{\chi_{j_1}(x,r)\ \left(\int \varphi(x,r) x^r\, dr\right)}{x^r\ \sum_{k=0}^{j_1} \tau_k^{j_1}\, (\log x)^{j_1-k}\, \varphi(x,r)^{(k)}}\right]^z\end{gathered} \tag{236}$$

Teorema 102 (Fórmula 2)**.** *Sea $u, v \in \mathbb{N}$, $r, z \in \mathbb{R}$, $r \geqslant 4$*

$$\varphi_r^z = \Xi_(1,r)^z \left[\frac{(2-\beta(r))^{u-1}\,(2+\beta(r))^v}{\Upsilon(r)^{u-1}\,\Gamma(r)^v}\right]^z \left(\int \varphi_r\, dr\right)^{(u+1)z} \left(\int \beta_r\, dr\right)^{vz} \tag{237}$$

Teorema 103. *Sea $j_1, j_2, k, u, \tau_k^j \in \mathbb{N}$, $x, r, z \in \mathbb{R}$, $z \geqslant 1$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\begin{gathered} \varphi_r^z \left[\frac{4^{u+1}\,(2-\beta(r))^{u-1}\,(2+\beta(r))^{u+1}}{\Upsilon(r)^u\,\Gamma(r)^{u+1}}\right]^z \left(\int\int \varphi_{2r}\, dr\, dr\right)^{(u+1)z} = \\ \left[\frac{\varphi_r^{(j_2)}}{\chi_{j_2}(1,r)}\right]^z \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^z \left[\frac{\chi_{j_1}(x,r)\,\left(\int \varphi(x,r) x^r dr\right)}{x^r\, \sum_{k=0}^{j_1} \tau_k^{j_1}\,(\log x)^{j_1-k}\,\varphi(x,r)^{(k)}}\right]^z \end{gathered} \tag{238}$$

Teorema 104 (Fórmula 3)**.** *Sea $u \in \mathbb{N}$, $r, z \in \mathbb{R}$, $r \geqslant 4$*

$$\varphi_r^z = \Xi_(1,r)^z \left[\frac{4^{u+1}\,(2-\beta(r))^{u-1}\,(2+\beta(r))^{u+1}}{\Upsilon(r)^{u-1}\,\Gamma(r)^{u+1}}\right]^z \left(\int\int \varphi_{2r}\, dr\, dr\right)^{(u+1)z} \tag{239}$$

Teorema 105. *Sea $j_1, j_2, k, n, u, \tau_k^j \in \mathbb{N}$, $x, r, z \in \mathbb{R}$, $z \geqslant 1$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\begin{gathered} \varphi_r^z \left[\frac{(2-\beta(r))^{u-1}}{\Upsilon(r)^u}\right]^z \left(\int \varphi_r\, dr\right)^{(u+1)z} = \\ \left[\frac{\varphi_r^{(j_2)}}{\chi_{j_2}(1,r,n)}\right]^z \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^z \left[\frac{\chi_{j_1}(x,r,n)\,\left(\int \varphi_n(x,r) x^r dr\right)}{x^r\, \sum_{k=0}^{j_1} \tau_k^{j_1}\,(\log x)^{j_1-k}\,\varphi_n(x,r)^{(k)}}\right]^z \end{gathered} \tag{240}$$

Teorema 106 (Fórmula 4)**.** *Sea $u, n \in \mathbb{N}$, $r, z \in \mathbb{R}$, $r \geqslant 4$*

$$\varphi_r^z = \Xi_(1,r,n)^z \left[\frac{\Upsilon_n(r)\,(2-\beta(r))^{u-1}}{\Upsilon(r)^u}\right]^z \left(\int \varphi_r\, dr\right)^{(u+1)z} \tag{241}$$

Teorema 107 (Fórmula 5)**.** *Sea $j, k, n, u \in \mathbb{N}$, $x, r, z \in \mathbb{R}$, $z \geqslant 1$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_r^z = \Xi_(1,r,n)^{\frac{z}{2}} \left[\frac{(2-\beta(r))^{u-1}}{\Upsilon(r)^u}\right]^{\frac{z}{2}} \left(\int \varphi_r\, dr\right)^{(u+1)\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{\frac{z}{2}} \left[\frac{\chi_j(x,r,n)\,\left(\int \varphi_n(x,r) x^r dr\right)}{x^r\, \sum_{k=0}^{j} \tau_k^{j}\,(\log x)^{j-k}\,\varphi_n(x,r)^{(k)}}\right]^{\frac{z}{2}} \tag{242}$$

Teorema 108 (Fórmula 6)**.** *Sea $k, u \in \mathbb{N}$, $x, r, z \in \mathbb{R}$, $z \geqslant 2$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_r^z = \Xi_(1,r)^{\frac{z}{2}} \left[\frac{4^{u+1}\,(2-\beta(r))^{u-1}\,(2+\beta(r))^{u+1}}{\Upsilon(r)^u\,\Gamma(r)^{u+1}}\right]^{\frac{z}{2}} \left(\int\int \varphi_{2r}\, dr\, dr\right)^{(u+1)\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{\frac{z}{2}} \left[\frac{\Xi(x,r)\,\left(\int \varphi(x,r) x^r dr\right)}{x^r\,\varphi(x,r)}\right]^{\frac{z}{2}} \tag{243}$$

Teorema 109 (Fórmula 7)**.** *Sea $k, u, v \in \mathbb{N}$, $x, r, z \in \mathbb{R}$, $z \geqslant 2$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\varphi_r^z = \Xi_(1,r)^{\frac{z}{2}} \left[\frac{(2-\beta(r))^{u-1}\,(2+\beta(r))^{v}}{\Upsilon(r)^u\,\Gamma(r)^{v}}\right]^{\frac{z}{2}} \left(\int \varphi_r\, dr\right)^{(u+1)\frac{z}{2}} \left(\int \beta_r\, dr\right)^{v\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{\frac{z}{2}} \left[\frac{\Xi(x,r)\,\left(\int \varphi(x,r) x^r dr\right)}{x^r\,\varphi(x,r)}\right]^{\frac{z}{2}} \tag{244}$$

Teorema 110 (Fórmula 8)**.** *Sea $k, u, v \in \mathbb{N}$, $x, r, z \in \mathbb{R}$, $z \geqslant 2$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\frac{1}{\varphi_r^z} = \left[\frac{\Upsilon(r)^u\,\Gamma(r)^v}{(2-\beta(r))^{u-1}\,(2+\beta(r))^{v}\,\Xi_(1,r)}\right]^{\frac{z}{2}} \left(\int \varphi_r\, dr\right)^{-(u+1)\frac{z}{2}} \left(\int \beta_r\, dr\right)^{-v\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk}\right)^{-\frac{z}{2}} \left[\frac{x^r\,\varphi(x,r)}{\Xi(x,r)\,\left(\int \varphi(x,r) x^r dr\right)}\right]^{\frac{z}{2}} \tag{245}$$

6.2 Función Zeta de Riemann

Teorema 111. *Sea $k, u, v \in \mathbb{N}$, $x, r, z \in \mathbb{R}$, $z \geqslant 2$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$*

$$\zeta_r(z) = 1 + \frac{1}{2^z} + \sum_{r=4}^{\infty} \frac{1}{\varphi_r^z} = 1 + \frac{1}{2^z} + \sum_{r=4}^{\infty} \left[\frac{\Upsilon(r)^u\, \Gamma(r)^v}{(2-\beta(r))^{u-1}\,(2+\beta(r))^v\, \Xi_(1,r)} \right]^{\frac{z}{2}} \left(\int \varphi_r\, dr \right)^{-(u+1)\frac{z}{2}} \left(\int \beta_r\, dr \right)^{-v\frac{z}{2}}$$
$$\left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{-\frac{z}{2}} \left[\frac{x^r\, \varphi(x,r)}{\Xi(x,r)\, \left(\int \varphi(x,r) x^r dr \right)} \right]^{\frac{z}{2}} \tag{246}$$

Demostración. Sea $x, r \in \mathbb{R}$, $r \geqslant 4$, $-1 < x < 1$, $x \neq \{0, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$.
Sea $r = 2^m$, donde $m \in \mathbb{N}$.
Y sea Θ una función en x y en r definida por

$$\Theta(x,r) = \left(\sum_{k=0}^{\infty} \frac{\varphi_{rk}}{\varphi_r} x^{rk} \right) = \frac{x^r}{x^{2r} - \beta(r) x^r + 1}$$

sabemos por el Límite A, Teorema 39 (83), que Θ converge cuando r es potencia de 2.
Queremos probar que $(\Theta)^{\frac{1}{2}}$ diverge, entonces haciendo r igual a 4 tenemos que

$$\left(\frac{x^4}{x^8 - 7x^4 + 1} \right)^{\frac{1}{2}} = \frac{x^2}{\sqrt{x^8 - 7x^4 + 1}},$$

supongamos que $(\Theta)^{\frac{1}{2}}$ converge, entonces por el teorema 32 (71) tendríamos que la siguiente serie

$$\left(\sum_{k=0}^{\infty} \varphi_{4k+2} x^{4k} \right)^{\frac{1}{2}} = \left(\frac{x^4 + 1}{x^8 - 7x^4 + 1} \right)^{\frac{1}{2}}$$

también converge y el grado de x en el numerador es 2, entonces usando el Corolario 9 (57) corresponde con la forma de la siguiente serie

$$\sum_{k=0}^{\infty} \varphi_{2k+1} x^{2k} = \frac{1 - x^2}{x^4 - 3x^2 + 1}$$

pero ésta lleva -1 en el coeficiente de x^2 en el numerador, entonces no existen los coeficientes para hacer converger la serie y $(\Theta)^{\frac{1}{2}}$ diverge. Por lo cual cuando z es 1 tenemos por el Teorema 110 (245) que ζ_r genera un 0, ya que la fórmula 8 se puede escribir y tenemos que

$$\left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{-\frac{z}{2}} = 0 \tag{247}$$

por tanto $\zeta_r(1) = 1 + \frac{1}{2} = \frac{3}{2}$.
Cuando z es $\frac{1}{2}$ tenemos que la suma diverge también ya que

$$\left(\sum_{k=0}^{\infty} \varphi_{rk} x^{rk} \right)^{-\frac{1}{4}} = 0,$$

por tanto para $z = \frac{1}{2}$

$$\zeta_r(z) = 1 + \frac{1}{\sqrt{2}}$$

y además para todo $z = \frac{1}{n}$, $n \in \mathbb{N}$, ζ_r cumple con

$$\zeta_r(z) = 1 + \frac{1}{\sqrt[n]{2}},$$

finalmente si $0 < z < 2$ entonces $\frac{z}{2} < 1$, la serie (247) diverge, el cociente es igual a 0 y $\zeta_r(z) = 1 + \frac{1}{2^z}$.
Por el contrario si $z \geqslant 2$ la serie (247) converge, el cociente no es 0 y ζ_r no genera ceros.

Sea $s \in \mathbb{R}$, $s \geqslant 4$, y tal que cumpla $2^{m-1} < s \leqslant 2^m$.
φ_s se puede escribir en la siguiente fórmula

$$\varphi_s\,\Theta(x,r) = \sum_{k=0}^{\infty} \varphi_s \frac{\varphi_{rk}}{\varphi_r} x^{rk}$$

entonces usando el Teorema 110 (245) podemos escribir la siguiente fórmula

$$\zeta_s(z) = 1 + \frac{1}{2^z} + \sum_{s=4}^{\infty} \frac{1}{\varphi_s^z} = 1 + \frac{1}{2^z} + \sum_{s=4}^{\infty} \left[\frac{\Upsilon(r)\,\Upsilon(s)^{u-1}\,\Gamma(s)^v}{(2-\beta(s))^{u-1}\,(2+\beta(s))^v\,\Xi_(1,s)} \right]^{\frac{z}{2}} \left(\int \varphi_s\,ds \right)^{-(u+1)\frac{z}{2}} \left(\int \beta_s\,ds \right)^{-v\frac{z}{2}} \left(\sum_{k=0}^{\infty} \varphi_s \frac{\varphi_{rk}}{\varphi_r} x^{rk} \right)^{-\frac{z}{2}} \left[\frac{\varphi(x,r)\,x^r}{\Xi(x,r)\,\left(\int \varphi(x,r) x^r dr \right)} \right]^{\frac{z}{2}} \tag{248}$$

y tenemos que para una cierta subsucesión $\alpha = \{\varphi_s\}$ se cumple con

$$\zeta_s^{\alpha}(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

■

Referencias

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